On the Separation of Maximally Violated mod-k Cuts

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Abstract. Separation is of fundamental importance in cutting-plane based techniques for *Integer Linear Programming* (ILP). In recent decades, a considerable research effort has been devoted to the definition of effective separation procedures for families of well-structured cuts. In this paper we address the separation of Chvátal rank-1 inequalities in the context of general ILP's of the form $\min\{c^T x : Ax \leq b, x \text{ integer}\}$, where A is an $m \times n$ integer matrix and b an m-dimensional integer vector. In particular, for any given integer k we study *mod-k* cuts of the form $\lambda^T Ax \leq \lfloor \lambda^T b \rfloor$ for any $\lambda \in \{0, 1/k, \ldots, (k-1)/k\}^m$ such that $\lambda^T A$ is integer. Following the line of research recently proposed for mod-2 cuts by Applegate, Bixby, Chvátal and Cook [1] and Fleischer and Tardos [16], we restrict to *maximally violated* cuts, i.e., to inequalities which are violated by $(k-1)/k$ by the given fractional point. We show that, for any given k, such a separation requires $O(mn \min\{m, n\})$ time. Applications to the TSP are discussed. In particular, for any given k, we propose an $O(|V|^2|E^*|)$-time exact separation algorithm for mod-k cuts which are maximally violated by a given fractional TSP solution with support graph $G^* = (V, E^*)$. This implies that we can identify a maximally violated TSP cut whenever a maximally violated *(extended) comb inequality* exists. Finally, specific classes of (sometimes new) facet-defining mod-k cuts for the TSP are analyzed.

1 Introduction

Separation is of fundamental importance in cutting-plane based techniques for *Integer Linear Programming* (ILP). In recent decades, a considerable research effort has been devoted to the definition of effective separation procedures for families of well-structured cuts. This line of research was originated by the pioneering work of Dantzig, Fulkerson and Johnson [12] on the *Traveling Salesman*

Problem (TSP) and led to the very successful branch-and-cut approach introduced by Padberg and Rinaldi [24]. Most of the known methods have been originally proposed for the TSP, a prototype in combinatorial optimization and integer programming.

In spite of the large research effort, however, polynomial-time exact separation procedures are known for only a few classes of facet-defining TSP cuts. In particular, no efficient separation procedure is known at present for the famous class of comb inequalities [19]. The only exact method is due to Carr [7], and requires $O(n^{2t+3})$ time for separation of comb inequalities with t teeth on a graph of n nodes. Recently, Letchford [21] proposed an $O(|V|^3)$-time separation procedure for a superclass of comb inequalities, applicable when the fractional point to be separated has a planar support.

Applegate, Bixby, Chvátal and Cook [1] recently suggested concentrating on maximally violated combs, i.e., on comb inequalities which are violated by 1/2 by the given fractional point x^* to be separated. This is motivated by the fact that maximally violated combs exhibit a very strong combinatorial structure, which can be exploited for separation. Their approach is heuristic in nature, and is based on the solution of a suitably-defined system of mod-2 congruences. Following this approach, Fleischer and Tardos [16] were able to design an $O(|V|^2 \log |V|)$-time exact separation procedure for maximally violated comb inequalities for the case where the support graph $G^* = (V, E^*)$ of the fractional point x^* is planar.

It is well known that comb inequalities can be obtained by adding-up and rounding a convenient set of TSP degree equations and subtour elimination constraints weighed by 1/2, i.e., they are $\{0, \frac{1}{2}\}$-*cuts* in the terminology of Caprara and Fischetti [6]. These authors studied $\{0, \frac{1}{2}\}$-cuts in the context of general ILP's. They showed that the associated separation problem is equivalent to the problem of finding a minimum-weight member of a binary clutter, i.e., a minimum-weight $\{0, 1\}$-vector satisfying a certain set of mod-2 congruences. This problem is NP-hard in general, as it subsumes the max-cut problem as a special case.

In this paper we address the separation of Chvátal rank-1 inequalities in the context of general ILP's of the form $\min\{c^T x : Ax \leq b, x \text{ integer}\}$, where A is an $m \times n$ integer matrix and b an m-dimensional integer vector. In particular, for any given integer k we study *mod-k* cuts of the form $\lambda^T Ax \leq \lfloor \lambda^T b \rfloor$ for any $\lambda \in \{0, 1/k, \ldots, (k-1)/k\}^m$ such that $\lambda^T A$ is integer. We show that, for any given k, separation of maximally violated mod-k cuts requires $O(mn \min\{m, n\})$ time as it is equivalent to finding a $\{0, 1, \ldots, k-1\}$-vector satisfying a certain set of mod-k congruences. We also discuss the separation of maximally violated mod-k cuts in the context of the TSP. In particular, we show how to separate efficiently maximally violated members of a family of cuts that properly contains comb inequalities. Interestingly, this family contains facet-inducing cuts which are not comb inequalities. We also show how to reduce from $O(|V|^2)$ to $O(|V|)$ the number of tight constraints to be considered in the mod-k congruence system, where $|V|$ is the number of nodes of the underlying graph. We investigate specific classes of (sometimes new) mod-k facet-defining cuts for the TSP and then give some concluding comments.

2 Maximally Violated mod-k Cuts

Given an $m \times n$ integer matrix A and an m-dimensional integer vector b, let $P := \{x \in R^n : Ax \leq b\}$, $P_I := \text{conv}\{x \in Z^n : Ax \leq b\}$, and assume $P_I \neq P$. A *Chvátal-Gomory cut* is a valid inequality for P_I of the form $\lambda^T Ax \leq \lfloor \lambda^T b \rfloor$, where the *multiplier vector* $\lambda \in R_+^m$ is such that $\lambda^T A \in Z^n$, and $\lfloor \cdot \rfloor$ denotes lower integer part. In this paper we address cuts which can be obtained through multiplier vectors λ belonging to $\{0, 1/k, \ldots, (k-1)/k\}^m$ for any given integer $k \geq 2$. We call them *mod-k cuts*, as their validity relies on mod-k rounding arguments. Note that mod-2 cuts are in fact the $\{0, \frac{1}{2}\}$-cuts studied in Caprara and Fischetti [6].

Any Chvátal-Gomory cut is a mod-k cut for some integer $k > 0$, as it is well known that undominated Chvátal-Gomory cuts only arise for $\lambda \in [0, 1)^m$, since replacing any λ_i by its fractional part $\lambda_i - \lfloor \lambda_i \rfloor$ always leads to an equivalent or stronger cut. Moreover, λ can always be assumed to be rational, i.e., an integer $k > 0$ exists such that $k\lambda$ is integer. Indeed, for any given $\lambda \in R_+^m$ with $\alpha^T := \lambda^T A \in Z^n$ one can obtain an equivalent (or better) multiplier vector $\tilde{\lambda}$ by solving the linear program $\min\{\tilde{\lambda}^T b : \tilde{\lambda}^T A = \alpha^T, \tilde{\lambda} \geq 0\}$, whose basic solutions are of the form $\tilde{\lambda} = \left[B^{-1}\alpha, 0\right]$ for some basis B of A^T. Hence $\det(B) \cdot \tilde{\lambda}$ is integer, as claimed.

We are interested in the following separation problem, in its optimization version:

mod-k SEP: *Given $x^* \in P$, find $\lambda \in \{0, 1/k, \ldots, (k-1)/k\}^m$ such that $\lambda^T A \in Z^n$, and $\lfloor \lambda^T b \rfloor - \lambda^T Ax^*$ is a minimum.*

Following [6], this problem can equivalently be restated in terms of the integer multiplier vector $\mu := k\lambda \in \{0, 1, \ldots, k-1\}^m$. For any given $z \in Z$ and $k \in Z_+$, let $z \bmod k := z - \lfloor z/k \rfloor k$. As is customary, notation $a \equiv b \pmod{k}$ stands for $a \bmod k = b \bmod k$. Given an integer matrix $Q = (q_{ij})$ and $k \in Z_+$, let $\overline{Q} = (\overline{q}_{ij}) := Q \bmod k$ denote the *mod-k support* of Q, where $\overline{q}_{ij} := q_{ij} \bmod k$ for all i, j. Then, mod-k SEP is equivalent to the following optimization problem.

mod-k SEP: *Given $x^* \in P$ and the associated slack vector $s^* := b - Ax^* \geq 0$, solve*

$$\delta^* := \min\ (s^{*\,T} \mu - \theta) \tag{1}$$

subject to

$$\overline{A}^T \mu \equiv 0 \pmod{k} \tag{2}$$

$$\overline{b}^T \mu \equiv \theta \pmod{k} \tag{3}$$

$$\mu \in \{0, 1, \ldots, k-1\}^m \quad (4)$$

$$\theta \in \{1, \ldots, k-1\}. \quad (5)$$

By construction, $(s^{*\,T}\mu - \theta)/k$ gives the slack of the mod-k cut $\lambda^T A x \leq \lfloor \lambda^T b \rfloor$ for $\lambda := \mu/k$, computed with respect to the given point x^*. Hence, there exists a mod-k cut violated by x^* if and only if the minimum δ^* in (1) is strictly less than 0. Observe that $s^* \geq 0$ and $\theta \leq k-1$ imply $\delta^* \geq 1-k$, i.e., no mod-k cut can be violated by more than $(k-1)/k$. This bound is attained for $\theta = k-1$, when the mod-k congruence system (2)–(4) has a solution μ with $\mu_i = 0$ whenever $s_i^* > 0$. In this case, the resulting mod-k cut is said to be *maximally violated*.

Even for $k = 2$, mod-k SEP is NP-hard as it is equivalent to finding a minimum-weight member of a binary clutter [6]. However, finding a *maximally violated* mod-k cut amounts to finding any feasible solution of the congruence system (2)–(4) after having fixed $\theta = k-1$ and having removed all the rows of $(\overline{A}, \overline{b})$ associated with a strictly positive slack s_i^*. For any k *prime* this solution, if any exists, can be found in $O(mn \min\{m, n\})$ time by standard Gaussian elimination in $GF(k)$.

For k nonprime $GF(k)$ is not a field, hence Gaussian elimination cannot be performed. On the other hand, there exists an $O(mn \min\{m, n\})$-time algorithm to find, if any, a solution of the mod-k congruence system (2)–(4) even for k nonprime, provided a prime factorization of k is known; see, e.g., Cohen [11].

The above considerations lead to the following result.

Theorem 1. *For any given k, maximally violated mod-k cuts can be found in $O(mn \min\{m, n\})$ time, provided a prime factorization of k is known.*

It is worth noting that mod-k SEP with $\mu_i = 0$ whenever $s_i^* > 0$ can be solved efficiently by fixing θ to any value in $\{1, \ldots, k-1\}$. We call the corresponding solutions of (2)–(5) *totally tight* mod-k cuts. The following theorem shows that, for k prime, the existence of a totally tight mod-k cut implies the existence of a maximally violated mod-k cut.

Theorem 2. *For any k prime, a maximally violated mod-k cut exists if and only if a totally tight mod-k cut exists.*

Proof. One direction is trivial, as a maximally violated mod-k cut is also a totally tight mod-k cut. Assume now that a totally tight mod-k cut exists, associated with a vector (μ, θ) satisfying (2)–(5) and such that $\mu_i = 0$ for all $s_i^* > 0$. If $\theta \neq k-1$ and k is prime, μ can always be scaled by a factor $w \in \{2, \ldots, k-1\}$ such that $\overline{A}^T w\mu \equiv 0 \ (\texttt{mod}\ k)$ and $\overline{b}^T w\mu \equiv k-1 \ (\texttt{mod}\ k)$.

Note that Theorem 2 cannot be extended to the case of k nonprime.

Of course, not all maximally violated mod-k cuts are guaranteed to be facet defining for P_I. In particular, a cut is not facet defining whenever it is associated with a nonminimal solution μ of the congruence system (2)–(4), where θ has been fixed to $k-1$ (barring the case of equivalent formulations of the same facet-defining cut). Indeed, the inequality associated with any solution $\tilde{\mu} \leq \mu$ is violated whenever the one associated with μ is. Hence one is motivated in finding maximally violated mod-k cuts which are associated with minimal solutions. This can be done with no extra computational effort for k prime since, for any fixed θ, all basic solutions to (2)–(4) are minimal by construction. Unfortunately, the algorithm for k nonprime does not guarantee finding a minimal solution. On the other hand, the following result holds.

Theorem 3. *If there exists a maximally violated mod-k cut for some k nonprime, a maximally violated mod-ℓ cut exists also for every ℓ which is a prime factor of k.*

Proof. First of all, observe that $Qy \equiv d \pmod{k}$ implies $Qy \equiv d \pmod{\ell}$ for each prime factor ℓ of k. Hence, given a solution (μ, θ) of (2)–(5) with $\theta = k - 1$, the vector $(\mu, \theta) \bmod \ell$ yields a totally tight mod-ℓ cut, as $\theta \bmod \ell = k - 1 \bmod \ell \neq 0$. The claim then follows from Theorem 2.

It is then natural to concentrate on the separation of maximally violated mod-k cuts for some k prime. For several important problems these cuts define facets of P_I, as shown for the TSP in Section 4.

3 Separation of Maximally Violated mod-*k* Cuts for the TSP

The TSP polytope is defined as the convex hull of the characteristic vectors of all the Hamiltonian cycles of a given complete undirected graph $G = (V, E)$. For any $S \subseteq V$, let $\delta(S)$ denote the set of the edges with exactly one end node in S, and $E(S)$ denote the set of the edges with both end nodes in S. Moreover, for any $A, B \in V$ we write $E(A : B)$ for $\delta(A) \cap \delta(B)$. As is customary, for singleton node sets we write v instead of $\{v\}$. For any real function $x : E \to R$ and for any $Q \subseteq E$, let $x(Q) := \sum_{e \in Q} x_e$.

A widely-used TSP formulation is based on the following constraints, called *degree equations*, *subtour elimination constraints (SEC's)*, and *nonnegativity constraints*, respectively:

$$x(\delta(v)) = 2, \ \text{ for all } v \in V \tag{6}$$

$$x(E(S)) \leq |S| - 1, \ \text{ for all } S \subset V, |S| \geq 2 \tag{7}$$

$$-x_e \leq 0, \ \text{ for all } e \in E. \tag{8}$$

We next address the separation of maximally violated mod-k cuts that can be obtained from (6)–(8). Given a point $x^* \in R^E$ satisfying (6)–(8), we call *tight* any node set S with $x^*(E(S)) = |S| - 1$. It is well known that only $O(|V|^2)$ tight sets exist, which can be represented by an $O(|V|)$-sized data structure called *cactus tree* [13]. A cactus tree associated with x^* can be found efficiently in $O(|E^*||V| \log(|V|^2/|E^*|))$, where $E^* := \{e \in E : x^*_e > 0\}$ is the support of x^*; see [15] and also [20]. Moreover, we next show that only $O(|V|)$ tight sets need be considered explicitly in the separation of maximally violated mod-k cuts.

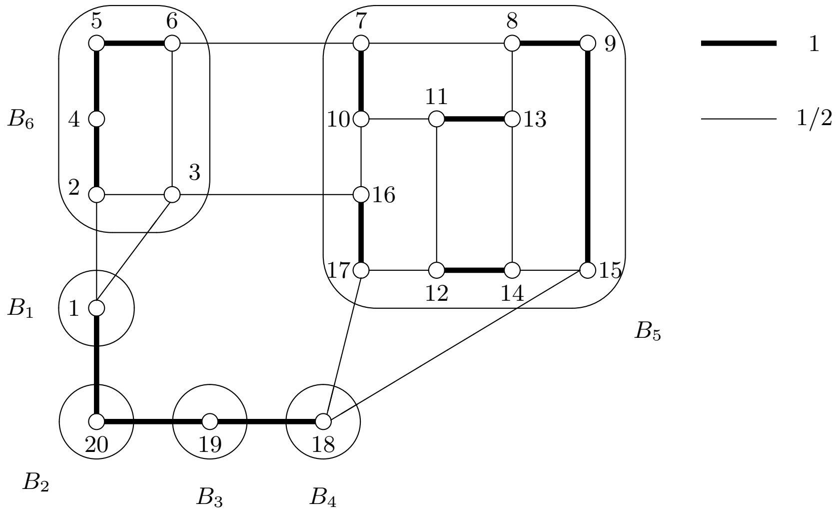


Fig. 1. A fractional point x^* and one of its necklaces.

Applegate, Bixby, Chvátal and Cook [1] and Fleischer and Tardos [16] showed that tight sets can be arranged in *necklaces*. A necklace of size $q \geq 3$ is a partition of V into a cyclic sequence of tight sets $B_1, \ldots, B_q$ called *beads*; see Figure 1 for an illustration. To simplify notation, the subscripts in $B_1, \ldots, B_q$ are intended modulo q, i.e., $B_i = B_{i+hq}$ for all integer h. Beads in a necklace satisfy:

(i) $B_i \cup B_{i+1} \cup \ldots \cup B_{i+t}$ is a tight set for all $i = 1, \ldots, q$ and $t = 0, \ldots, q-2$,
(ii) $x^*(E(B_i : B_j))$ is equal to 1 if $j \in \{i+1, i-1\}$, and 0 otherwise.

A pair (B_i, B_{i+1}) of consecutive beads in a necklace is called a *domino*. We allow for degenerate necklaces with $q = 2$ beads, in which $x^*(E(B_1 : B_2)) = 2$. Degenerate necklaces have no dominoes.

Given x^* satisfying (6)–(8), one can find in time $O(|E^*||V| \log(|V|^2/|E^*|))$ a family $\mathcal{F}(x^*)$ of $O(|V|)$ necklaces with the property that every tight set is the union of consecutive beads in a necklace of the family. The next theorem shows that the columns in the congruence system (2)–(4) corresponding to tight SEC's are linearly dependent, in $GF(k)$, on a set of columns associated with degree equations, tight nonnegativity constraints, and tight SEC's corresponding to beads and dominoes in $\mathcal{F}(x^*)$.

Theorem 4. *If any TSP mod-k cut is maximally violated by x^*, then there exists a maximally violated mod-k cut whose Chvátal-Gomory derivation uses SEC's associated with beads and dominoes (only) of necklaces of $\mathcal{F}(x^*)$.*

Proof. Let S be any tight set whose SEC is used in the Chvátal-Gomory derivation of some maximally violated mod-k cut. By the properties of $\mathcal{F}(x^*)$, S is the union of consecutive beads $B_1, \ldots, B_t$ of a certain necklace $B_1, \ldots, B_q$ in $\mathcal{F}(x^*)$, $1 \leq t \leq q-1$. If $t \leq 2$, then S is either a bead or a domino, and there is nothing to prove. Assume then $t \geq 3$, as in Figure 2, and add together:

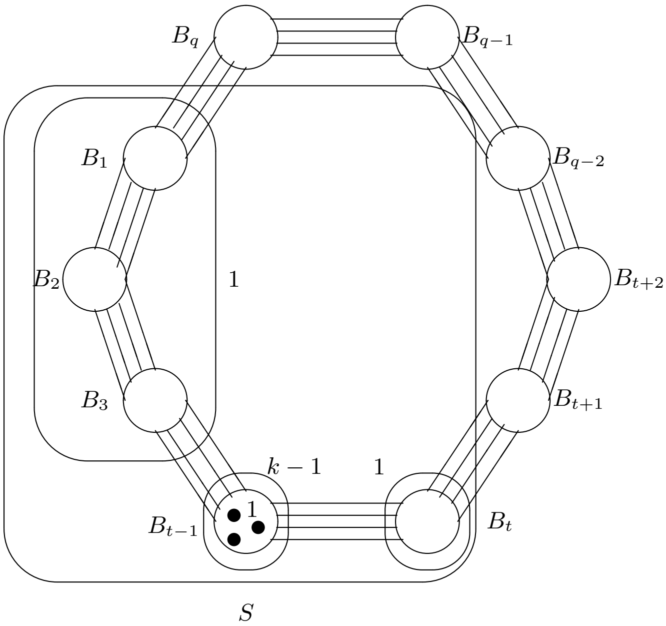


Fig. 2. Illustration for the proof of Theorem 4.

- the SEC on $B_1 \cup B_2 \cup \ldots \cup B_{t-2}$,
- the SEC on B_{t-1} multiplied by $k-1$,
- the SEC on B_t,
- the degree equations on every $v \in B_{t-1}$,
- the nonnegativity inequalities $-x_e \leq 0$ for every $e \in E(B_{t-1} : B_{t+1} \cup \ldots \cup B_q)$,
- the nonnegativity inequalities $-x_e \leq 0$ multiplied by $k-1$ for every $e \in E(B_t : B_1 \cup \ldots \cup B_{t-2})$.

This gives the following inequality:

$$\alpha^T x := x(E(S)) + kx(E(B_{t-1})) - kx(E(B_t : B_1 \cup \ldots \cup B_{t-2})) \leq$$
$$\alpha_0 := |S| + k|B_{t-1}| - k - 1.$$

All the inequalities used in the combination are tight at x^*. Moreover, all the coefficients in $\alpha^T x \leq \alpha_0$ are identical, modulo k, to the coefficients of the SEC $x(E(S)) \leq |S| - 1$. So we can use the inequalities in the derivation of $\alpha^T x \leq \alpha_0$ in place of the original SEC to obtain a (different) maximally violated mod-k cut. Applying this procedure recursively yields the result.

As an immediate consequence, one has

Theorem 5. *For any given k, maximally violated mod-k cuts for the TSP can be found in $O(|E^*||V|^2)$ time, i.e., in $O(|V|^4)$ time in the worst case.*

Proof. Theorem 1 gives an $O(mn \min\{m, n\})$-time separation algorithm, where m is the number of tight constraints (6)-(7), and $n = |E^*| = O(|V|^2)$ is the number of fractional components in x^*. By virtue of Theorem 4, only $O(|V|)$ tight sets need be considered explicitly, hence $m = O(|V|)$ and the claim follows.

The practical efficiency of the mod-k separation algorithm can be improved even further, as it turns out that one can always disregard all dominoes except one (arbitrarily chosen) in each necklace. This is expressed in the following theorem, whose proof is contained in the full paper.

Theorem 6. *Let $\mathcal{B}$ contain, for each necklace in $\mathcal{F}(x^*)$, all beads and a single (arbitrary) domino. If any TSP mod-k cut is maximally violated by x^*, then there exists a maximally violated mod-k cut whose Chvátal-Gomory derivation uses SEC's associated with sets $S \in \mathcal{B}$ only.*

In the full paper we will also derive similar results for the Asymmetric TSP polytope.

4 Specific Classes of mod-*k* Cuts for the TSP

In this section we analyze specific classes of facet-defining mod-k cuts for the Symmetric TSP. We also briefly mention some analogous results for the Asymmetric TSP which will be presented in detail in the full paper.

We first address mod-2 cuts that can be obtained from (6)–(8). A well known class of such cuts is that of *comb* inequalities, as introduced by Edmonds [14] in the context of matching theory, and extended by Chvátal [10] and by Grötschel and Padberg [17, 18] for the TSP. Comb inequalities are defined as follows. We are given a *handle* set $H \subset V$ and $t \geq 3$, t odd, *tooth* sets $T_1, \dots, T_t \subset V$ such that $T_i \cap H \neq \emptyset$ and $T_i \setminus H \neq \emptyset$ hold for any $i = 1, \dots, t$. The comb inequality associated with $H, T_1, \dots, T_t$ reads:

$$x(E(H)) + \sum_{i=1}^{t} x(E(T_i)) \leq |H| + \sum_{i=1}^{t}(|T_i| - 1) - \frac{t+1}{2}. \tag{9}$$

The simplest case of comb inequalities arises for $|T_i| = 2$ for $i = 1, \dots, t$, leading to the Edmonds' *2-matching constraints*. It is well known that comb inequalities define facets of the TSP polytope [19]. Also well known is that comb inequalities are mod-2 cuts.

As already mentioned, no polynomial-time exact separation algorithm for comb inequalities is known at present. A heuristic scheme for maximally violated comb inequalities has been recently proposed by Applegate, Bixby, Chvátal and Cook [1], and elaborated by Fleischer and Tardos [16] to give a polynomial-time exact method for the case of x^* with planar support. Here, comb separation is viewed as the problem of "building-up" a comb structure starting with a given set of dominoes. The interested reader is referred to [1] and [16] for a detailed description of the method.

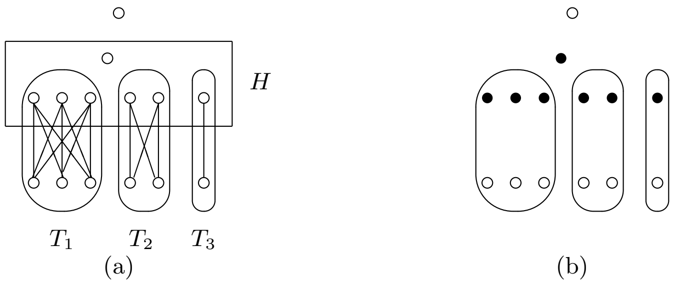


Fig. 3. (a) The support graph of a simple extended comb inequality; all the drawn edges, as well as the edges in $E(H)$, have coefficient 1. (b) A mod-2 derivation, obtained by combining the degree equations on the black nodes and the SEC's on the sets drawn in continuous line (the nonnegativity inequalities used in the derivation are not indicated).

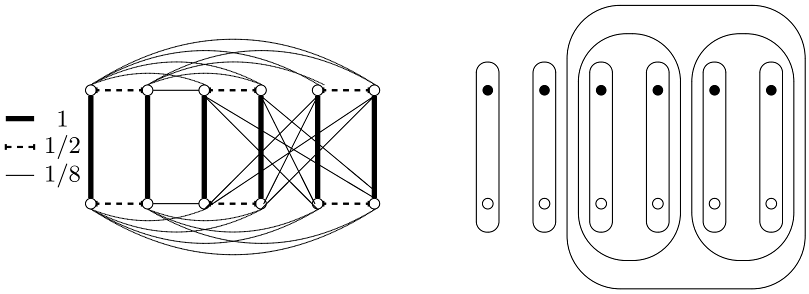
Theorem 5 puts comb separation in a different light, in that it allows for efficient exact separation of maximally violated members of the family of mod-2 cuts which contains, among others, comb inequalities.

One may wonder whether comb inequalities are the only TSP facet-defining mod-2 cuts with respect to formulation (6)–(8). This is not the case; in particular, we address in the full paper the facet-defining *extended comb* inequalities of Naddef and Rinaldi [22] (see Figure 3 for an illustration) and prove the following

Theorem 7. *Extended comb inequalities are facet-defining TSP mod-2 cuts.*

Extended comb inequalities can be derived from 2-matching constraints by means of two general lifting operations, called *edge-cloning* and *0-node lifting.* These operations have been studied by Naddef and Rinaldi [23] who proved that, under mild assumptions, they preserve the facet-defining property of the original inequality. Interestingly, at least for the case of extended comb inequalities both operations do not increase the Chvátal rank [25] of the starting inequality, and also preserve the property of being a mod-2 cut. One may wonder whether this property is true in general. An answer to this question will be given in the full paper, where we study the two operations in the more general context of the Asymmetric TSP.

A family $\mathcal{N}$ of sets $S_1, \ldots, S_k \subseteq V$ is called *nested* (or *laminar*) if, for all i, j, $S_i \cap S_j \neq \emptyset$ implies $S_i \subseteq S_j$ or $S_j \subseteq S_i$. The node sets associated with SEC's with nonzero multipliers in the Chvátal-Gomory derivation of an extended comb inequality define a nested family $\mathcal{N}$ with nesting degree not greater than 2, in the sense that $\mathcal{N}$ does not contain 3 subsets $S_1 \subset S_2 \subset S_3$. Actually, it is easy to show that any mod-k cut can be derived by only using SEC's associated with subsets defining a nested family. Interestingly, there are mod-2 facet-defining TSP cuts whose Chvátal-Gomory derivation involves SEC's with nesting level greater than 2. Here is an example. Consider the fractional point x^* of Figure 4(a). It is not



(a) (b)

Fig. 4. (a) A fractional point x^* which violates maximally no extended comb inequality. (b) The derivation of a mod-2 cut which is maximally violated by x^*.

hard to check by complete enumeration that x^* maximally violates no extended comb inequality. However, x^* maximally violates the mod-2 cut whose derivation is illustrated in Figure 4(b). It can be shown that this inequality is facet defining for the TSP polytope when $|V| \geq 12$.

Examples of facet-defining mod-3 cuts for the TSP are the Christof, Jünger and Reinelt [9] NEW1 inequality along with a generalization which we will introduce in the full paper.

Also in the full paper we give examples of facet-defining mod-k cuts for the Asymmetric TSP. These include, for $k = 2$, a generalization of the *source-destination* inequalities of Balas and Fischetti [5], for $k = 3$, generalizations of the NEW1 inequality and the *C3* inequalities of Grötschel and Padberg [19] and, finally, for arbitrary k, generalizations of the D_k^+ and D_k^- inequalities of Grötschel and Padberg [19].

5 Concluding Remarks

Recent developments in cutting-plane algorithms, such as the work of Balas, Ceria and Cornuéjols [2, 3] and Balas, Ceria, Cornuéjols and Natraj [4] on lift-and-project (disjunctive) cuts and Gomory cuts, put the emphasis on the separation of large classes of inequalities which are not given explicitly. The approach developed in this paper provides still another tool for tackling hard problems.

Future theoretical research should be devoted to the study of the structure of undominated (facet-defining) mod-k TSP cuts. One should also address mod-k cuts for other combinatorial problems. Furthermore, the practical use of the separation methods herein proposed should be investigated.

References

1. D. Applegate, R. Bixby, V. Chvátal, W. Cook (1995). Finding cuts in the TSP (A preliminary report). Technical Report Technical Report 95–05, DIMACS, Rutgers University, New Brunswick, NJ.
2. E. Balas, S. Ceria, G. Cornuéjols (1993). A lift-and-project cutting plane algorithm for mixed 0-1 programs. *Math. Program. (A)* 58, 295–324.
3. E. Balas, S. Ceria, G. Cornuéjols (1996). Mixed 0-1 programming by lift-and-project in a branch-and-cut framework. *Management Sci.* 42, 1229–1246.
4. E. Balas, S. Ceria, G. Cornuéjols, N. Natraj (1996). Gomory cuts revisited. *Oper. Res. Lett.* 19, 1–9.
5. E. Balas, M. Fischetti (1993). A lifting procedure for the asymmetric traveling salesman polytope and a large new class of facets. *Math. Program. (A)* 58, 325–352.
6. A. Caprara, M. Fischetti (1996). $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts. *Math. Program. (A)* 74, 221–235.
7. R. Carr (1995). Separating clique tree and bipartition inequalities in polynomial time. E. Balas, J. Clausen (eds.). *Integer Programming and Combinatorial Optimization 4*, Lecture Notes in Computer Science, 920, Berlin. Springer-Verlag, 40–49.
8. S. Ceria, G. Cornuéjols, M. Dawande (1995). Combining and strengthening Gomory cuts. E. Balas, J. Clausen (eds.). *Integer Programming and Combinatorial Optimization 4*, Lecture Notes in Computer Science, 920, Berlin. Springer-Verlag, 438–451.
9. T. Christof, M. Jünger, G. Reinelt (1991). A complete description of the traveling salesman polytope on 8 nodes. *Oper. Res. Lett.* 10, 497–500.
10. V. Chvátal (1973). Edmonds polytopes and weakly Hamiltonian graphs. *Math. Program.* 5, 29–40.
11. H. Cohen (1995). *A Course in Computational Algebraic Number Theory*, Springer-Verlag, Berlin.
12. G. Dantzig, D. Fulkerson, S. Johnson (1954). Solution of a large scale traveling-salesman problem. *Oper. Res.* 2, 393–410.
13. E.A. Dinitz, A.V. Karzanov, M.V. Lomosonov (1976). On the structure of a family of minimal weighted cuts in a graph. A.A. Fridman (ed.) *Studies in Discrete Optimization*, Moscow Nauka, 290–306 (in Russian).
14. J. Edmonds (1965). Maximum matching and a polyhedron with 0,1-vertices. *J. Res. Natl. Bureau of Standards* 69, 125–130.
15. L. Fleischer (1998). Building the chain and cactus representations of all minimum cuts from Hao-Orlin in same asymptotic run time. R. Bixby, E. Boyd, R. Rios Mercado (eds.). *Integer Programming and Combinatorial Optimization 6*, Lecture Notes in Computer Science, Berlin. Springer-Verlag.
16. L. Fleischer, É. Tardos (1996). Separating maximally violated comb inequalities in planar graphs. W. Cunningham, S. McCormick, M. Queyranne (eds.). *Integer Programming and Combinatorial Optimization 5*, Lecture Notes in Computer Science, 1084, Berlin. Springer-Verlag, 475–489. Revised version to appear in *Math. Oper. Res.*
17. M. Grötschel, M. Padberg (1979). On the symmetric traveling salesman problem I: Inequalities. *Math. Program.* 16, 265–280.
18. M. Grötschel, M. Padberg (1979). On the symmetric traveling salesman problem II: lifting theorems and facets. *Math. Program.* 16, 281–302.

19. M. Grötschel, M. Padberg (1985). Polyhedral theory. E. Lawler, J. Lenstra, A. Rinnooy Kan, D. Shmoys (eds.). *The Traveling Salesman Problem*, John Wiley & Sons, Chichester, 251–305.
20. D. Karger, C. Stein (1996). A new approach to the minimum cut problem. *J. ACM* 43, 601–640.
21. A.N. Letchford (1998). Separating a superclass of comb inequalities in planar graphs. Technical Report, Dept. of Man. Science, The Management School, Lancaster University, 1998.
22. D. Naddef, G. Rinaldi (1988). The symmetric traveling salesman polytope: New facets from the graphical relaxation. Technical Report 248, IASI-CNR, Rome.
23. D. Naddef, G. Rinaldi (1993). The graphical relaxation: A new framework for the symmetric traveling salesman polytope. *Math. Program. (A)* 58, 53–88.
24. M. Padberg, G. Rinaldi (1991). A branch and cut algorithm for the resolution of large-scale symmetric traveling salesman problems. *SIAM Rev.* 33, 60–100.
25. A. Schrijver (1986). *Theory of Linear and Integer Programming*, John Wiley & Sons, New York.