Chapter 6

POLYHEDRAL THEORY FOR ARC ROUTING PROBLEMS

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1. INTRODUCTION

As explained in Chapter 4, most realistic Arc Routing Problems are known to be $\mathcal{NP}$-hard. Therefore we can expect that there will be certain instances which are impossible to solve to optimality within a reasonable time. However, this does not mean that *all* instances will be impossible to solve. It may well be that an instance which arises in practice has some structure which makes it amenable to solution by an optimization algorithm. Since, in addition, significant costs are often involved in real-world instances, research into devising optimization algorithms is still regarded as important.

At the time of writing, the most promising optimization algorithms for $\mathcal{NP}$-hard problems are based on the so-called *branch-and-cut* method (see Padberg & Rinaldi, 1991 and also Chapter 7 of this book). The key to producing an effective branch-and-cut algorithm for a particular class of problems is to have a good understanding of certain polyhedra which are associated with those problems.

This chapter gives a self-contained introduction to the main ideas of polyhedral theory, followed by a state-of-the-art survey of the known polyhedral results for Arc Routing. The most promising integer programming formulations of various Arc Routing Problems are reviewed and the known valid inequalities and facets of the associated polyhedra are presented. Note, however, that we are not here concerned with the *algorithmic* implications of polyhedral theory. These are discussed in detail in Chapter 7.

The outline of the chapter is as follows. In Section 2, the basic ideas and notation involved in polyhedral theory are summarised. In Section 3, the definitions of a number of routing problems are briefly reviewed. The following two sections review the polyhedral theory for single-vehicle variants of the Chinese Postman Problem and the Rural Postman Problem, respectively. Section 6 does the same for the Capacitated Arc Routing Problem. Some concluding comments are made in Section 7.

Other surveys about Arc Routing which include material on polyhedral theory can be found in Assad & Golden (1995) and Eiselt et al. (1995a, b).

2. THE BASICS OF POLYHEDRAL THEORY

This section draws on material in Weyl (1935), Grötschel & Padberg (1985) and Nemhauser & Wolsey (1988).

Like many other combinatorial optimization problems, the majority of Arc Routing Problems can be formulated as problems of the form:

$$\min\{c^T x : x \in \mathcal{S}\} \tag{6.1}$$

where $x = \{x_1, \ldots, x_n\} \in \Re^n$ is a vector of decision variables, $c = \{c_1, \ldots, c_n\} \in \Re^n$ is a vector of objective function coefficients (i.e., costs) and $\mathcal{S} \subset \Re^n$ is a set of *feasible solutions*. Often, but not always, $\mathcal{S}$ is defined by an explicit system of linear inequalities with integer coefficients, together with an integrality condition. That is, there is an integer m, a matrix $A \in Z^{mn}$ and a vector $b \in Z^m$ such that

$$\mathcal{S} = \{x \in Z^n : Ax \leq b\}. \tag{6.2}$$

In this case, (6.1) is called an *Integer Linear Program* (ILP). Note that constraints of the 'greater than or equal to' form can be accommodated in (6.2) if they are multiplied by minus one; equations can also be accommodated since they can be written as two inequalities. Note also that m may be very large; often it is exponential in n.

The key to the polyhedral approach is the observation that the feasible solutions to (6.1) (i.e., the members of $\mathcal{S}$) are vectors in the Euclidean space $\Re^n$. It is then easy to define a polyhedron associated with a given $\mathcal{S}$. In order to show how this can be done, it is first necessary to give some formal definitions.

A set $H \subset \Re^n$ is called a *half-space* if there exists a vector $a \in \Re^n$ and a scalar $a_0 \in \Re$ such that $H = \{x \in \Re^n : a^T x \leq a_0\}$. A set $P \subset \Re^n$ is called a *polyhedron* if it is the intersection of finitely many half-spaces. A polyhedron which is bounded (i.e., not of infinite volume) is called a *polytope*. Now suppose that $x^1, \ldots, x^k \in \Re^n$ are vectors and $\lambda_1, \ldots, \lambda_k$ are scalars. A vector of the form $\lambda_1 x^1 + \cdots + \lambda_k x^k$ is called an *affine combination* of $x^1, \ldots, x^k$ if $\sum_{i=1}^k \lambda_i = 1$. An affine combination is also called a *convex* combination if $\lambda_i \geq 0$ for all i. Given some $\mathcal{S} \subset \Re^n$ with $\mathcal{S} \neq \emptyset$, the *affine* (respectively, *convex*) *hull* of $\mathcal{S}$, denoted by aff($\mathcal{S}$) (respectively, conv($\mathcal{S}$)), is the set of all affine (convex) combinations of finitely many vectors in $\mathcal{S}$.

Note that for any $\mathcal{S}$, conv($\mathcal{S}$) $\subseteq$ aff($\mathcal{S}$) holds. Also, aff($\mathcal{S}$) is always a polyhedron. The situation is a little more complicated for conv($\mathcal{S}$). It can be shown that, when $|\mathcal{S}|$ is finite, conv($\mathcal{S}$) is always a polyhedron. When $|\mathcal{S}|$ is not finite, however, conv($\mathcal{S}$) can fail to be a polyhedron (see, e.g., Queyranne & Wang, 1992). Fortunately, however, the sets $\mathcal{S}$ which will be of concern to us in this chapter are 'well-behaved', in that conv($\mathcal{S}$) will always be a polyhedron.

Another important concept is that of the *dimension* of a polyhedron. This is defined using the idea of *affine independence*. A set of vectors is affinely independent if no member of the set is an affine combination of the others. The dimension of a polyhedron P, denoted by dim(P), is then defined as the maximum number of affinely independent vectors in P. Note that, if P is defined in $\Re^n$, then dim(P) $\leq n$ holds. If equality holds, P is said to be *full-dimensional*.

Given these definitions, we can now associate a polyhedron with any instance of a combinatorial optimization problem; namely, the polyhedron conv($\mathcal{S}$), where $\mathcal{S}$ is the set of feasible solutions to (6.1). Because the vectors in $\mathcal{S}$ are assumed to be integral, conv($\mathcal{S}$) will be a polyhedron with integral vertices. We will let P_I denote such an integral polyhedron. The whole aim of the polyhedral approach is to find 'good' descriptions of P_I for various combinatorial optimization problems. In order to define what is meant by a 'good' description, we need some further definitions.

An inequality $a^T x \leq a_0$ is *valid* for P_I if $P_I \subseteq \{x \in \Re^n : a^T x \leq a_0\}$. The set $F = P_I \cap \{x \in \Re^n : a^T x = a_0\}$ is called the *face induced by* $a^T x \leq a_0$ *on* P_I. Note that F is also a polyhedron. A valid inequality is *supporting* if $F \neq \emptyset$. If $F = P_I$, then $a^T x = a_0$ is said to be an *implicit equation* of P_I. The face F of P_I induced by a valid inequality $a^T x \leq a_0$ is called a *facet* if $F \neq P_I$ and there is no other valid inequality which induces a face F' of P_I such that F is strictly contained in F'. Note that $\dim(F) = \dim(P_I)$ - 1 when F is a facet.

A 'best possible' complete linear description of P_I must therefore include all implicit equations and facet-inducing inequalities. Of course, if there are implicit equations, then any one of the equations or inequalities can be written in an infinite number of ways. All that is really needed, however, is one representative of each non-equivalent facet-inducing inequality, along with a minimal set of equations which identifies the affine hull of P_I.

When an instance of an $\mathcal{NP}$-hard combinatorial optimization problem is extremely small, it may be possible to find such a *complete linear description* of the associated P_I. However, finding a complete linear description of P_I for instances of realistic size is hopelessly difficult. Although the number of equations needed to identify the affine hull is small (only n - $\dim(P_I)$), it often happens that P_I has an immense number of facets. An example will drive this point home. Readers who are familiar with the well-known *Symmetric Travelling Salesman Problem* (STSP) will regard an STSP instance defined on a graph with only 10 vertices as trivial. Nevertheless, Christof & Reinelt (1996) have shown that over 5.1 x 10^{10} inequalities are needed to describe P_I in this case (together with 10 equations, one for each vertex in the graph).

A further negative result comes from Karp & Papadimitriou (1982), who showed that, if a combinatorial optimization problem is $\mathcal{NP}$-hard, then the problem of deciding whether or not an inequality is valid for the associated P_I is itself $\mathcal{NP}$-hard.

Despite these negative results, however, it remains true that large classes of valid, supporting and even facet-inducing inequalities, along with implicit equations, are known for many important $\mathcal{NP}$-hard problems. This will become clear in the following subsections. Such *partial* linear descriptions of polyhedra provide the basis for very effective optimization algorithms, see Chapter 7.

3. THE ROUTING PROBLEMS DEFINED

In this section, we give the definitions of twelve different fundamental routing problems for which polyhedral studies have been conducted. These will be examined in subsequent sections.

First, we consider undirected, single-vehicle problems. Given a connected, undirected graph G with vertex set V and (undirected) edge set E, a cost c_e for each edge $e \in E$, a set $V_R \subseteq V$ of *required vertices* and a set $E_R \subseteq E$ of *required edges*, the *General Routing Problem* (GRP) is the problem of finding a minimum cost vehicle route ('tour') passing through each $v \in V_R$ and each $e \in E_R$ at least once (Orloff, 1974).

The GRP contains a number of other known problems as special cases. When $E_R = \emptyset$, the GRP reduces to the *Steiner Graphical Travelling Salesman Problem* (SGTSP) (Cornuéjols et al., 1985), also called the *Road Travelling Salesman Problem* by Fleischmann (1985). On the other hand, when $V_R = \emptyset$, the GRP reduces to the *Rural Postman Problem* (RPP) (Orloff, 1974). When $V_R = V$, the SGTSP in turn reduces to the *Graphical Travelling Salesman Problem* or GTSP (Cornuéjols et al., 1985). Similarly, when $E_R = E$, the RPP reduces to the *Chinese Postman Problem* or CPP (Guan, 1962; Edmonds, 1963).

The CPP can be solved in polynomial time by reduction to a matching problem (Christofides, 1973; Edmonds & Johnson, 1973), but the RPP, GTSP, SGTSP and GRP are all $\mathcal{NP}$-hard. The GTSP and SGTSP were proved to be $\mathcal{NP}$-hard by Cornuéjols et al. (1985) and the RPP and GRP were proved to be $\mathcal{NP}$-hard by Lenstra & Rinnooy-Kan (1976).

Now we consider problems in which directed arcs are allowed. Given a connected, mixed graph G with vertex set V, (undirected) edge set E, (directed) arc set A, a cost c_e for each edge $e \in E$, a cost c_a for each arc $a \in A$, a set $E_R \in E$ of *required edges* and a set $A_R \in A$ of *required arcs*, the *Mixed Rural Postman Problem* (MRPP) is the problem of finding a minimum cost vehicle route passing through each $e \in E_R$ and each $a \in A_R$ at least once (Corberán, Romero & Sanchis, 1997; Romero, 1997).

Like the GRP, the MRPP also contains a number of other problems as special cases. When $A = \emptyset$, the MRPP reduces to the RPP mentioned above. When $E = \emptyset$, the MRPP reduces to the *Directed Rural Postman Problem* or DRPP (Christofides et al., 1986). Alternatively, it may be that $E_R = E$ and $A_R = A$, in which case the MRPP reduces to the *Mixed Chinese Postman Problem* or MCPP (Edmonds & Johnson, 1973). When $E_R = \emptyset$, the MCPP in turn reduces to the *Directed Chinese Postman Problem* or DCPP (Edmonds & Johnson, 1973).

The DCPP can be solved in polynomial time by reduction to a transportation problem (Edmonds & Johnson, 1973), but the MRPP, DRPP and MCPP are all $\mathcal{NP}$-hard. The MCPP (and therefore MRPP) was proved to be $\mathcal{NP}$-hard by Papadimitriou (1976) and the DRPP was proved to be $\mathcal{NP}$-hard by Christofides et al. (1986).

Another single-vehicle problem is known as the *Windy Postman Problem* or WPP. It is a generalisation of the CPP which allows for the possibility that the cost of traversing an edge in one direction may differ from the cost of traversing the edge in the opposite direction. The WPP combines features of both undirected and directed problems. In fact, it is easy to show that it contains the MCPP as a special case. It is therefore $\mathcal{NP}$-hard, as noted by Guan (1984), although it is polynomially solvable in certain special cases (Guan, 1984; Win, 1987, 1989).

Now we consider two undirected multi-vehicle problems. The *Capacitated Arc Routing Problem* or CARP is a generalization of the RPP in which $k \geq 1$ identical vehicles are available, each of *capacity* $Q > 0$. One particular vertex is called the *depot* and each required edge has an integral *demand* $q_i \geq 0$. The task is to find a minimum cost set of k feasible routes, each one starting and ending at the depot. (A route is feasible if the sum of the demands on the route do not exceed Q.) When $E_R = E$, the CARP reduces to the *Capacitated Chinese Postman Problem* or CCPP (Win, 1987).

Since the CARP is at least as difficult as the RPP, it is $\mathcal{NP}$-hard (Golden & Wong, 1981). Perhaps more surprisingly, the CCPP is also $\mathcal{NP}$-hard. In fact, Golden & Wong (1981) showed that it is $\mathcal{NP}$-hard to find a *0.5-approximate* solution to the CCPP (i.e., a solution which has a cost less than 1.5 times the cost of the optimal solution).

4. VARIANTS OF THE CHINESE POSTMAN PROBLEM

4.1. THE CPP

Since the CPP can be solved in polynomial time, we might expect that the associated polyhedra have a simple description. This is indeed the case. We let the general integer variable x_e represent the number of times that edge e is traversed without being serviced. (That is, x_e represents the number of copies of edge e which will be added to E in order to make G Eulerian.) For each $S \subset V$, we let $\delta(S)$ denote the set of edges, commonly called the *edge cutset*, which have one end-vertex in S and one end-vertex in $V \setminus S$. When $S = \{i\}$, we write $\delta(i)$ rather than $\delta(\{i\})$ for brevity. Finally, for any $F \subset E$, we let $x(F)$ denote $\sum_{e \in F} x_e$. Then our set $\mathcal{S}$ of feasible solutions is defined as

$$x(\delta(i)) \equiv |\delta(i)| \pmod 2 (\forall i \in V) \tag{6.3}$$

$$x_e \geq 0 (\forall e \in E) \tag{6.4}$$

$$x \in Z^{|E|} \tag{6.5}$$

Note that the conditions (6.3) are *not* in the form of linear inequalities. They do however imply the validity of the following *odd-cut* (or *blossom*) inequalities (Edmonds & Johnson, 1973):

$$x(\delta(S)) \geq 1 \quad (\forall S \subset V : |\delta(S)| \text{ odd}). \tag{6.6}$$

To see why these inequalities are valid, note that the vehicle must cross any given edge cutset an even number of times. Hence, if $S \subset V$ is such that $|\delta(S)|$ is odd, then the vehicle must cross the cutset at least once without servicing. Note that the number of odd-cut inequalities can increase exponentially in the size of the graph G.

Edmonds & Johnson (1973) show that P_I is completely described by the odd-cut inequalities (6.6) and the *non-negativity* inequalities (6.4). That is, the conditions (6.3) and (6.5) are unnecessary.

Corberán & Sanchis (1994) have shown that an odd-cut inequality is facet-inducing if and only if the subgraphs induced in G by S and $V \setminus S$ are each connected; they also showed that a non-negativity inequality induces a facet if and only if e is not a *cut-edge* (an edge whose removal disconnects the graph).

4.2. THE DCPP

Since the DCPP can be solved in polynomial time, we might expect that the associated polyhedra have a simple description, just as in the case of the CPP. This is indeed the case. We let the general integer variable x_a represent the number of times that arc a is traversed without being serviced. This is analogous to the formulation for the CPP given in the previous subsection, in that x_a represents the number of copies of arc a which will be added to A in order to make G Eulerian. For any $i \in V$, we let $\delta^+(i)$ denote the set of arcs which leave i and $\delta^-(i)$ denote the set of arcs which enter i. Finally, we let $b(i)$ for each $i \in V$ denote $|\delta^-(i)| - |\delta^+(i)|$, the so-called 'unbalance' of i. Note that $b(i)$ may be positive, negative or zero. The set $\mathcal{S}$ is then given by (Edmonds & Johnson, 1973):

$$x(\delta^+(i)) - x(\delta^-(i)) = b(i)(\forall i \in V) \tag{6.7}$$

$$x_a \geq 0(\forall a \in A) \tag{6.8}$$

$$x \in Z^{|A|} \tag{6.9}$$

Equations (6.7), which we will call *balance* equations, ensure that the vehicle leaves each vertex as many times as it enters. They describe the affine hull of P_I and in fact it can be shown that only $|V| - 1$ of them are needed (that is, any one of them can be deleted as redundant). It is also not hard to show that the non-negativity inequalities (6.8) induce facets of P_I under mild conditions.

Edmonds & Johnson (1973) showed that P_I is completely described by the balance equations and non-negativity inequalities. Thus, the integrality condition (6.9) is unnecessary.

4.3. THE MCPP

When considering how to tackle the MCPP, it appears at first sight that in addition to an integer variable for each arc, it will be necessary to have *two* integer variables for each edge (to indicate the number of times the edge is traversed in either direction). Indeed, formulations of this kind have appeared in the literature (e.g., Kappauf & Koehler, 1979; Christofides et al., 1984; Grötschel & Win, 1992; Ralphs, 1993). However, it is possible to use only one variable per edge (Nobert & Picard, 1996), as we now explain.

Each MCPP solution is defined by a family (i.e., a set with possible repeated elements) of edges and arcs which constitute what might be called a *mixed Eulerian multigraph*. In Ford & Fulkerson (1962), it is proven that a mixed multigraph is Eulerian if and only if it is *even* and *balanced*. The first condition means that the number of arcs and edges incident on any vertex is an even integer. To explain the second condition, we extend our notation a little. Given any $S \subset V$, let $\delta(S)$ be the set of all *edges* crossing >from S to $V \setminus S$, $\delta^+(S)$ the set of all *arcs leaving* S, and let $\delta^-(S)$ be the set of all *arcs entering* S. The condition that the multigraph be *balanced* means that, for any $S \subset V$, $|\delta^-(S)| - |\delta^+(S)| \leq |\delta(S)|$ holds. Ford & Fulkerson also provide a simple algorithm to find a tour of a mixed Eulerian multigraph.

Now let the general integer variable x_e (respectively, x_a) represent the number of times that edge e (respectively, arc a) is traversed without being serviced. That is, each variable represents the number of times a particular edge or arc will be added to $E \cup A$ to make G Eulerian. Also, for any $S \subset V$ let $b(S) = |\delta^-(S)| - |\delta^+(S)| - |\delta(S)|$, the so-called 'unbalance' of S. Note that this definition of 'unbalance' is a generalization of the definition given in the previous subsection. Finally, let $\delta^*(S)$ denote $\delta(S) \cup \delta^+(S) \cup \delta^-(S)$. Then the set $\mathcal{S}$ of feasible solutions is defined by the following conditions:

$$x(\delta^*(i)) \equiv |\delta^*(i)| \pmod 2 (\forall i \in V) \tag{6.10}$$

$$x(\delta^+(S)) + x(\delta(S)) - x(\delta^-(S)) \geq b(S) (\forall S \subset V) \tag{6.11}$$

$$x_e \geq 0 (\forall e \in E) \tag{6.12}$$

$$x_a \geq 0 (\forall a \in A) \tag{6.13}$$

$$x \in Z^{|E \cup A|} \tag{6.14}$$

The system of congruences (6.10) enforces the condition that the associated mixed multigraph be even. Similarly, the *balanced set* inequalities (6.11) enforce the condition that the multigraph be balanced. Finally, (6.12) and (6.13) are just non-negativity inequalities. Obviously, the inequalities (6.11) - (6.13) are valid for the associated P_I.

It might be thought that a 'switched' version of (6.11) would also be needed for each S, i.e., an inequality of the form $x(\delta^-(S)) - x(\delta^+(S)) + x(\delta(S)) \geq -b(S)$. However, this is easily shown to be equivalent to the balanced set inequality associated with $V \setminus S$. Using this fact, it is also possible to show that P_I is not full-dimensional in general (though this is not noted explicitly by Nobert & Picard). Suppose that $S \subset V$ is such that $|\delta(S)| = \emptyset$. Then, the balanced set inequality reduces to $x(\delta^+(S)) - x(\delta^-(S)) \geq b(S)$, whereas, the balanced set inequality for $V \setminus S$ can be written as $x(\delta^+(S)) - x(\delta^-(S)) \leq b(S)$. Thus, $x(\delta^+(S)) - x(\delta^-(S)) = b(S)$ is an implicit equation of P_I. We could call equations of this type *balance equations*. It is easily shown that they are a generalization of the balance equations (6.7) for the DCPP.

One further class of valid inequalities is presented in Nobert & Picard (1996). They note that the condition (6.10) implies that the following *blossom* inequalities are valid for P_I:

$$x(\delta^*(S)) \geq 1 \quad (\forall S \subset V : |\delta^*(S)| \text{ odd}). \tag{6.15}$$

These are a simple generalization of the odd-cut (blossom) inequalities for the CPP.

The algorithm of Nobert & Picard (1996), based upon the formulation given here, clearly outperformed algorithms based upon formulations with two variables per edge.

In Subsection 5.4, we review a study by Corberán, Romero & Sanchis (1997) about polyhedra associated with the Mixed Rural Postman Problem (MRPP). Since the MRPP contains the MCPP as a special case, many of the results of Corberán, Romero & Sanchis also apply to the MCPP. This implies, for example, that the affine hull of P_I is described by one balance equation for each connected component in the subgraph of G induced by the (required) edges and that exactly one of these equations is redundant. It also yields necessary and sufficient conditions for the non-negativity, blossom and balanced set inequalities to induce facets of P_I. The details are not given here, for brevity.

We would like to close this subsection with a question for future research:

Research Problem: For what (mixed) graphs G (if any) does the polyhedron defined by (6.11) - (6.15) contain integer extreme points which do *not* represent feasible tours?

4.4. THE WPP

Although the WPP is notionally defined on an undirected graph, we will assume that the underlying graph is directed, with two arcs (i, j), (j, i) going in opposite directions for each edge $e = \{i, j\}$ in the original graph. Then, we can define a general integer variable x_{ij} for each arc in the directed graph, representing the number of times the vehicle travels in that particular direction. The set $\mathcal{S}$ of feasible solutions is then given by:

$$x(\delta^+(i)) - x(\delta^-(i)) = 0 (\forall i \in V) \quad (6.16)$$

$$x_{ij} + x_{ji} \geq 1 (\forall \{i, j\} \in E) \quad (6.17)$$

$$x_{ij}, x_{ji} \geq 0 (\forall \{i, j\} \in E) \quad (6.18)$$

$$x \in Z^{2|E|} \quad (6.19)$$

The associated polyhedron is examined in Win (1987) and Grötschel & Win (1988). They show that the *balance equations* (6.16) describe the affine hull of P_I and that only $|V| - 1$ of them are needed (any one of them can be deleted as redundant). They also show that the inequalities (6.17), which ensure that each edge is traversed at least once, together with the non-negativity inequalities (6.18), are facet-inducing.

Grötschel and Win also show that the following *odd-cut* inequalities are valid and that they induce facets under mild conditions:

$$x(\delta^+(S)) + x(\delta^-(S)) \geq |\delta(S)| + 1 \quad (\forall S \subset V : |\delta(S)| \text{ odd}). \tag{6.20}$$

They also mention that the odd-cut inequality for any S can be re-written, using the balance equations, in a variety of other forms, such as:

$$x(\delta^+(S)) \geq \frac{1}{2}(|\delta(S)| + 1)$$

or

$$x(\delta^-(S)) \geq \frac{1}{2}(|\delta(S)| + 1).$$

We close this section by mentioning a result of Win (1987, 1989), who showed that the polyhedron defined by (6.16), (6.17) and (6.18) is *half-integral*. That is, all of its extreme points have components that are an integral multiple of one-half. He also showed that this polyhedron is integral if and only if the original graph is Eulerian. Ralphs (1993) gave a similar result for an analogous formulation for the MCPP.

5. VARIANTS OF THE RURAL POSTMAN PROBLEM

5.1. THE RPP

An integer programming formulation for the RPP was given in Christofides et al. (1981), but the associated polyhedron was not examined in detail. This was done in Corberán & Sanchis (1994). In the Corberán & Sanchis (1994) formulation, x_e represents the number of times e is traversed (if $e \notin E_R$), or one less than this number (if $e \in E_R$). That is, x_e represents the number of copies of e which will be added to E_R in order to form an Eulerian multigraph. Now we let V_R denote the set of vertices incident on at least one required edge (these vertices are also 'required' since the vehicle must travel through them) and let $\delta_R(S)$ denote $\delta(S) \cap E_R$. The set $\mathcal{S}$ is then defined by

$$x(\delta(S)) \geq 2, (\forall S \subset V : \delta_R(S) = \emptyset, S \cap V_R \neq \emptyset, V_R \setminus S \neq \emptyset) \tag{6.21}$$

$$x(\delta(i)) \equiv |\delta_R(i)| \pmod 2, \forall i \in V \tag{6.22}$$

$$x_e \geq 0 (\forall e \in E) \tag{6.23}$$

$$x \in Z^{|E|} \tag{6.24}$$

Corberán & Sanchis (1994) show that the associated polyhedron is full-dimensional and unbounded. Many classes of valid inequalities and facets are known. In this subsection, we outline those presented in Corberán & Sanchis (1994). The other known inequalities were discovered in the context of the *General Routing Problem* and will be described in the next subsection.

Constraints (6.21), called *connectivity inequalities*, ensure that the route is connected. They induce facets of P_I if and only if the subgraphs induced in G by S and $V \setminus S$ are connected. The non-negativity inequalities (6.23) induce facets if and only if e is not a cut-edge. Another simple class of valid inequalities is given by the following *R-odd cut* inequalities:

$$x(\delta(S)) \geq 1 \quad (\forall S \subset V : |\delta_R(S)| \text{ odd}) \tag{6.25}$$

These generalise the odd-cut (blossom) inequalities for the CPP and are valid for the same reason. Like connectivity inequalities, R-odd cut inequalities induce facets if and only if the subgraphs induced in G by S and $V \setminus S$ are each connected (Corberán & Sanchis, 1994).

In order to present the remaining inequalities, we will need some more definitions. Consider the (generally disconnected) subgraph of G obtained by deleting all non-required edges from G. We call a connected component of this subgraph an *R-component*. Also, given two disjoint subsets A and B of V, we let $E(A : B)$ denote the set of edges in E with one end-vertex in A and one in B and $E_R(A : B)$ denote $E(A : B) \cap E_R$.

A *K-component* (K-C) *configuration* is a partition $\{V_0, \ldots V_K\}$ of V, with $K \geq 3$, such that

- $V_1, \ldots V_{K-1}$ and $V_0 \cup V_K$ are clusters of node sets of one or more R-components,
- $|E_R(V_0 : V_K)|$ is positive and even,
- $E(V_i : V_{i+1}) \neq \emptyset$ for $i = 0, \ldots, K-1$.

Figure 1 shows a K-C configuration: the filled circles represent the V_i, the bold lines represent the required edges crossing from V_0 to V_K and the plain lines represent the non-required edges which must be present. Associated with a K-C configuration is a *K-C inequality*, which can be written as:

$$\sum_{p=0}^{K-1} \sum_{q=p+1}^{K} (q-p)x(E(V_p : V_q)) - 2x(E(V_0 : V_K)) \geq 2(K-1) \tag{6.26}$$

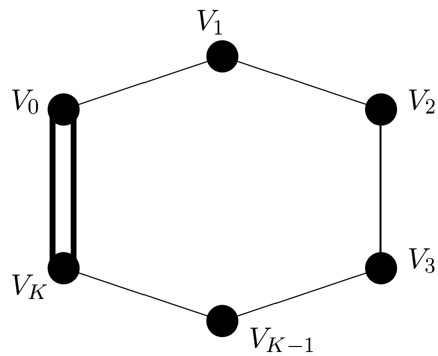

Figure 6.1 K-C configuration.

K-C inequalities induce facets when certain mild connectivity assumptions are met (Corberán & Sanchis, 1994).

The last class of inequalities presented in Corberán & Sanchis (1994) are known as *GTSP-type* inequalities. Suppose we have an RPP instance and let m be the number of R-components in G. Consider a partition of V into sets $S_1, \dots, S_m$ such that $S_i \cap V_R \neq \emptyset$ and $\delta_R(S_i) = \emptyset$ for $i = 1, \dots, m$. Let G_s be the (multi)graph obtained from G by shrinking each S_i into a single vertex and eliminating loops. That is, G_s has m vertices. It is now possible to define a GTSP instance on G_s and Corberán & Sanchis (1994) show that any (non-trivial) facet-inducing inequality for the resulting GTSP polyhedron is also facet-inducing for the polyhedron associated with the original RPP instance.

This is a powerful result, because GTSP polyhedra have been widely studied and many classes of facet-inducing inequalities are known for them. Space does not permit a review here, so the reader is directed to the surveys Jünger, Reinelt & Rinaldi (1995, 1997).

We close this subsection by mentioning some recent results due to Ghiani & Laporte (1997). Suppose that $V = V_R$ (a problem in which $V \neq V_R$ can be easily transformed into one in which $V = V_R$, see e.g., Christofides et al., 1981). Consider the graph obtained by 'shrinking' each R-component down to a single vertex (but *not* merging parallel edges). Find a minimum cost spanning tree T on this shrunk graph. Then Ghiani & Laporte show that there exists an optimal RPP solution such that $x_e \leq 2$ for each $e \in T$ and $x_e \leq 1$ for each $e \notin T$. This leads to a pure 0-1 formulation for the RPP, with one binary variable for each edge, where each $e \in T$ is split into two parallel edges with the same cost.

For the Ghiani & Laporte formulation, P_I is a polytope (that is, a bounded polyhedron). This polytope is rather different from the unbounded polyhedron studied by Corberán & Sanchis. For one thing, it is typically not full-dimensional (this is easily shown by considering sets $S \subset V$ with $1 \leq |\delta(S)| \leq 3$). For another, there are new valid inequalities. For example, let $S \subset V$ and $F \subset \delta(S)$ be such that $|\delta_R(S)| + |F|$ is odd. Then the inequalities

$$x(\delta(S) \setminus F) \geq x(F) - |F| + 1 \tag{6.27}$$

are valid for the polytope but not for the unbounded polyhedron. These inequalities proved useful in the branch-and-cut algorithm of Ghiani & Laporte.

It is not immediately obvious which approach to the RPP is 'best', whether that of Corberán & Sanchis or that of Ghiani & Laporte. The authors regard this as an interesting theoretical and empirical problem. Of course, any facet-inducing inequalities for the unbounded polyhedron are also valid for the polytope. However, they cannot be guaranteed to induce facets any longer.

5.2. THE GRP

When tackling the GRP, it is helpful to assume (w.l.o.g.) that the end-vertices of each required edge are in V_R. Define for each $e \in E$ a general integer variable x_e, representing the number of times e is traversed (if $e \notin E_R$), or one less than this number (if $e \in E_R$). Then, the system (6.21) - (6.23) given in the last section defines $\mathcal{S}$ in the case of the GRP as well as in the special case of the RPP. Also, the connectivity, R-odd cut, K-C and GTSP-type inequalities are valid for the GRP as well as the RPP (Corberán & Sanchis, 1998). A number of new results are also known for the GRP and are presented in the remainder of this subsection. It should be noted that these results are also new even when specialised to the RPP.

In Corberán & Sanchis (1998), the K-C inequalities were generalised to give the *honeycomb* inequalities, which also define facets if certain mild connectivity assumptions are met. A *honeycomb configuration* is a partition of V into sets S_i such that:

- for all i, $| \delta(S_i) \setminus \delta_R(S_i) | \neq \emptyset$ and $| \delta_R(S_i) |$ is even or zero;
- there are at least two values i such that $\delta_R(S_i) \neq \emptyset$;
- there are at least two values i such that $\delta_R(S_i) = \emptyset$;

- there is a set T of non-required edges crossing between the S_i forming a tree spanning the S_i, such that each member of T crosses between sets S_i and S_j with $E_R(S_i : S_j) = \emptyset$.

Many, but not all, honeycomb configurations can be formed by 'gluing' K-C configurations together by identifying edges (Corberán & Sanchis, 1998).

The coefficients in the associated honeycomb inequality are defined as follows, apart from one exception mentioned in the next paragraph. Let α_e denote the coefficient of x_e in the honeycomb inequality. Then $\alpha_e = 0$ if and only if $e \notin \delta(S_i)$ for all i and $\alpha_e = 1$ for all $e \in T$. The coefficient α_e of any other crossing non-required edge e is equal to the number of edges traversed in T to get from one end-vertex of e to the other. For the required edges crossing between the S_i, the coefficient is 2 units less.

The exception is that, for certain complex honeycomb configurations, some of the crossing non-required edges which are not in T may have a smaller coefficient. In such cases, the α_e must be computed sequentially (by a so-called *sequential lifting* procedure).

The honeycomb inequality is then:

$$\sum_{e \in E} c_e x_e \geq 2(K-1) \tag{6.28}$$

Figure 2 shows two honeycomb configurations. The bold lines represent edges in $\delta_R(V_i)$ for some i and the plain lines represent edges in the spanning tree. In the corresponding inequalities, all edges shown have a coefficient equal to 1. The rhs is 6 in both cases.

In Letchford (1997a), the facet-inducing *path-bridge* (PB) inequalities were introduced. Like honeycomb inequalities, PB inequalities are a generalization of K-C inequalities. However, they are a generalization in a different direction. They are defined in terms of an associated *path-bridge* (PB) *configuration.* Suppose $p \geq 1$ and $b \geq 0$ are integers such that $p + b \geq 3$ and odd. Let $n_i \geq 2$ for $i = 1, \ldots, p$ also be integers. A PB configuration is (see Figure 3) a partition of V into vertex sets A, Z and V_j^i for $i = 1, \ldots, p$, $j = 1, \ldots, n_i$ with the following properties:

- each V_j^i is a cluster of one or more R-sets,
- $|E_R(A : Z)| = b$,
- $E(A : V_1^i) \neq \emptyset$ and $E(V_{n_i}^i : Z) \neq \emptyset$ for $i = 1, \ldots, p$,
- $E(V_j^i : V_{j+1}^i) \neq \emptyset$ for $i = 1, \ldots, p$ and $j = 1, ..., n_i - 1$.

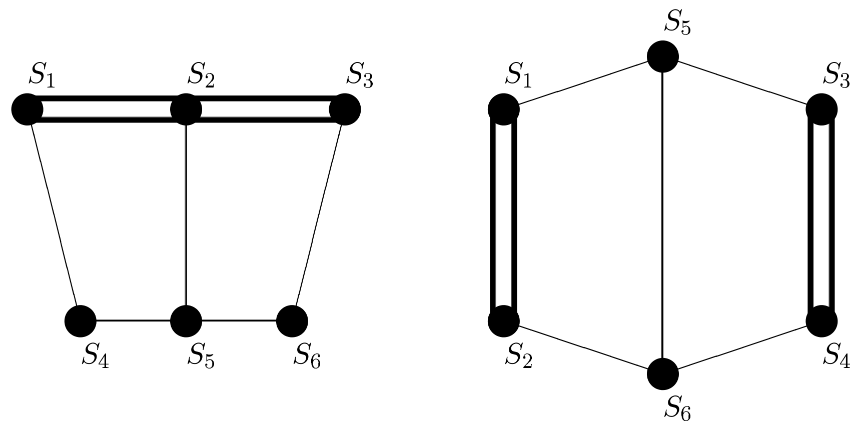

Figure 6.2 Two honeycomb configurations.

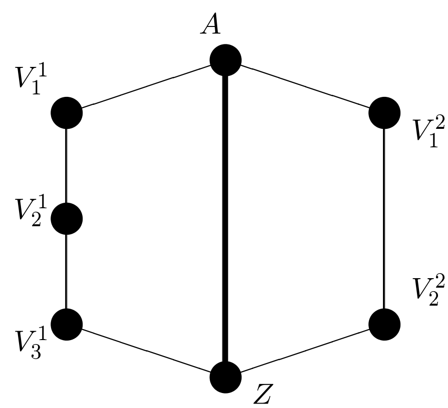

Figure 6.3 PB configuration.

The edges in $E_R(A:Z)$ constitute the *bridge*. If the bridge is empty ($b = 0$), either or both of A and Z are permitted to be empty also.

To define the coefficients of the associated PB inequality, it is helpful to identify A with V_0^i and Z with $V_{n_i+1}^i$ for $i = 1, \ldots, p$. The PB inequality is then:

$$\sum_{e \in E} \alpha_e x_e \geq 1 + \sum_{i=1}^{p} \frac{n_i + 1}{n_i - 1} \tag{6.29}$$

where the coefficient α_e for an edge $e = \{u, v\}$ is defined as follows. If $u \in V_j^i$ and $v \in V_k^i$, $j \geq k$, then $\alpha_e = (j-k)/(n_i - 1)$; unless $u \in A$ and $v \in Z$, in which case $\alpha_e = 1$. If, however, $u \in V_j^i$ and $v \in V_k^r$, with $i \neq r$,

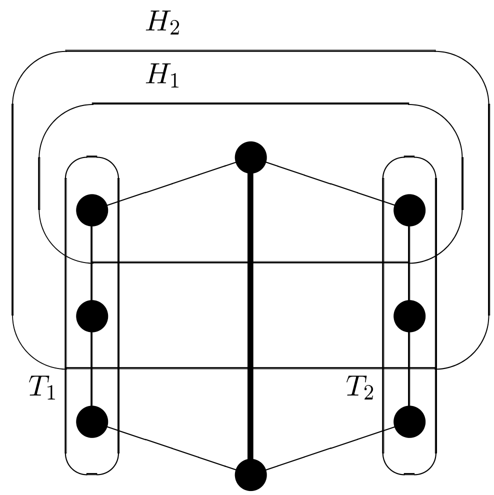

Figure 6.4 Handles and teeth in a 3-regular PB configuration.

$1 \leq j \leq n_i$, $1 \leq k \leq n_r$, then α_e equals

$$\frac{1}{n_i - 1} + \frac{1}{n_r - 1} + |\frac{j-1}{n_i - 1} - \frac{k-1}{n_r - 1}|.$$

The PB inequalities include many other known inequalities as special cases. When the bridge is empty ($b = 0$), we have the *path* inequalities of Cornuéjols, Fonlupt & Naddef (1985), valid for the STSP and GTSP. When $p = 1$, the PB inequalities reduce to K-C inequalities.

Another special case of interest is when all the n_i for $i = 1, \dots, p$ are equal to a same value n. In such cases, the PB inequality is called n-*regular* (a term applied by Cornuéjols et al., 1985, to path inequalities). Note that K-C inequalities can be regarded as 'degenerate' n-regular PB inequalities, with $n = K + 1$. The n-regular PB inequalities have a nice description in terms of vertex sets called *handles* and *teeth*. There are $n-1$ handles, denoted by $H_1, \dots, H_{n-1}$, and p teeth, denoted by $T_1, \dots, T_p$ (see Figure 4). The first handle is defined as $H_1 = A \cup V_1^1 \cup \ldots \cup V_1^p$; the other handles are defined inductively as $H_i = H_{i-1} \cup V_i^1 \cup \ldots \cup V_i^p$. The teeth are defined as $T_j = V_1^j \cup \ldots \cup V_n^j$. The n-regular PB inequality is then:

$$\sum_{i=1}^{n-1} x(\delta(H_i)) + \sum_{j=1}^{p} x(\delta(T_j)) \ \geq \ np + n + p - 1. \qquad (6.30)$$

As shown by Cornuéjols, Fonlupt & Naddef (1985), the 2-regular path inequalities are equivalent to the well-known *comb* inequalities for the

STSP, which in turn include the well-known *2-matching* inequalities as a special case (see, e.g., Grötschel & Padberg, 1985). Thus, the 2-regular PB inequalities can also be regarded as a generalization of the comb and 2-matching inequalities.

Since both the GTSP-type inequalities reviewed in the previous section and the PB inequalities mentioned above are analogous to facets of GTSP polyhedra, it might be suspected that there is some general procedure for adapting polyhedral results for the GTSP into results for the GRP. This is indeed the case. In Letchford (1997b), it is shown how to generalize valid or facet-inducing inequalities for the GTSP to the more general SGTSP (defined in Section 3). Then, it is shown that the polyhedron P_I for any given GRP instance is a face of a polyhedron of an associated SGTSP instance. Finally, a certain *projection operation* is given which enables any class of valid inequalities for the GTSP to be adapted to the GRP. We do not describe the arguments involved in any detail for the sake of brevity; instead, we only mention a few key results:

- R-odd cut inequalities are *projected 2-matching* inequalities;
- K-C and PB inequalities are *projected path* inequalities;
- many of the honeycomb inequalities are *projected binested* inequalities.

5.3. THE DRPP

In this subsection we will need to adapt the notation somewhat. For a given $S \subset V$, we define $\delta^+(S)$, $\delta^-(S)$ and $\delta^*(S)$ as in Section 4. In addition, $A_R(S)$ will denote the required arcs with both end-vertices in S. We also set $\delta^+_R(S) = \delta^+(S) \cap A_R$, $\delta^-_R(S) = \delta^-(S) \cap A_R$ and $\delta^*_R(S) = \delta^*_R(S) \cup \delta^-_R(S)$. Finally, $b(i)$ for each $i \in V$ will denote $|\delta^-_R(i)| - |\delta^+_R(i)|$. By analogy with the DCPP (Subsection 4.2) and the MCPP (Subsection 4.3), $b(i)$ can be thought of as the 'unbalance' of vertex i.

The natural approach to the DRPP is to have a general integer variable x_a for each arc $a \in A$, representing the number of times the arc is traversed without being serviced (see Christofides et al., 1986; Ball & Magazine, 1988). Again, this means that x_a represents the number of copies of a that will be added to A_R to form an Eulerian graph. This leads to the following definition of the set $\mathcal{S}$ of feasible solutions:

$$x(\delta^+(i)) - x(\delta^-(i)) = b(i)(\forall i \in V) \tag{6.31}$$

$$x(\delta^+(S)) \geq 1(\forall S \subset V : \begin{cases} A_R(S) \neq \emptyset, \\ A_R(V \setminus S) \neq \emptyset, \\ \delta^*_R(S) = \emptyset) \end{cases} \tag{6.32}$$

$$x_a \geq 0(\forall a \in A) \tag{6.33}$$

$$x \in Z^{|A|} \tag{6.34}$$

The associated P_I were studied independently by Savall (1990) and Gun (1993). As for the DCPP and WPP, the *balance* equations (6.31) define the affine hull of P_I and precisely $|V|-1$ of them are independent. The *postman cut* inequalities (6.32) are the directed analogue of the connectivity inequalities (6.21) for the RPP. Surprisingly, however, the conditions for them to induce facets are extremely complicated (there is not space to describe them here). Finally, the non-negativity inequalities (6.33) induce facets under mild conditions.

5.4. THE MRPP

Because the MRPP is a common generalization of the RPP, DRPP and MCPP, all of which have a complicated enough polyhedral structure, it will be clear to the reader that MRPP polyhedra are likely to be bewilderingly complicated (indeed, even the notation becomes burdensome). Nevertheless, a study has recently been made by Corberán, Romero & Sanchis (1997) and Romero (1997). To simplify the study, these authors assume that

- every $v \in V$ is incident on at least one required edge or arc

- $E = E_R$.

Instances which do not meet these assumptions are transformed by a simple procedure into instances which do.

Just as in all of the formulations examined so far (apart from the one for the WPP in Subsection 4.4), Corberán, Romero & Sanchis let x_e (or x_a) represent the total number of times that an edge (or arc) is traversed without being serviced. They define $\delta^-(S)$, $\delta^+(S)$, $\delta(S)$, $\delta^*(S)$, $\delta^-_R(S)$, $\delta^+_R(S)$, etc., as in previous subsections. Finally, they define an 'unbalance' $b(S) = |\delta^-_R(S)| - |\delta^+_R(S)| - |\delta_R(S)|$ for each $S \subset V$. The set

$\mathcal{S}$ of feasible solutions is then given by:

$$x(\delta^*(i)) \equiv |\delta^*(i)| \pmod 2 (\forall i \in V) \tag{6.35}$$

$$x(\delta^+(S)) \geq 1 (\forall S \subset V : \delta^*_R(S) = \emptyset) \tag{6.36}$$

$$x(\delta^+(S)) + x(\delta(S)) - x(\delta^-(S)) \geq b(S) (\forall S \subset V) \tag{6.37}$$

$$x_e \geq 0 (\forall e \in E) \tag{6.38}$$

$$x_a \geq 0 (\forall a \in A) \tag{6.39}$$

$$x \in Z^{|E \cup A|} \tag{6.40}$$

The *connectivity*, *balanced set* and *trivial* inequalities, (6.36), (6.37) and (6.38 - 6.39), respectively, are valid for the associated polyhedron P_I. Note that when $\delta(S) = \emptyset$, the balanced set inequalities for S and $V \setminus S$ imply the equation

$$x(\delta^+(S)) - x(\delta^-(S)) = b(S). \tag{6.41}$$

Corberán, Romero & Sanchis show that the affine hull of P_I is described by one such equation for each connected component in the subgraph of G induced by the (required) edges and that exactly one of these equations is redundant.

A mixed graph is *strongly connected* if and only if there is a path from any vertex to any other vertex, respecting the directions of the arcs. Note that G must be strongly connected for the MRPP to have a feasible solution. Corberán, Romero & Sanchis show that:

- A connectivity inequality induces a facet if and only if the subgraphs induced in G by S and $V \setminus S$ are both strongly connected (otherwise, they are dominated by balanced set inequalities).
- A trivial inequality induces a facet if and only if the subgraph formed by removing the corresponding edge (or arc) from G is strongly connected.
- The odd-cut (blossom) inequalities for the CPP, MCPP and RPP can be adapted to the MRPP, where they take the form:

$$x(\delta^*(S)) \geq 1 \quad (\forall S \subset V : |\delta^*_R(S)| \text{ odd}) \tag{6.42}$$

The conditions for the balanced set and odd-cut inequalities to induce facets are rather complicated and will not be given here.

Finally, the same authors adapt the K-C and PB inequalities for the RPP (see Subsections 5.1 and 5.2) to the MRPP. Interestingly, they show that these inequalities come in two distinct (non-equivalent) ‘flavours’ in the mixed case. One version has the same coefficients as in the ordinary RPP, but the other has slightly different coefficients. Both versions induce distinct facets under certain (complicated) conditions.

6. THE CAPACITATED ARC ROUTING PROBLEM

6.1. PRELIMINARIES

In this section, we present the known polyhedral results for the CARP. Most of the results presented are applicable also for the CCPP, which is a special case of the CARP, except that some valid inequalities are only defined when there is more than one R-component. Since Golden & Wong (1981) proved that it is $\mathcal{NP}$-hard even to find a 0.5-approximate solution to the CCPP (see also Win, 1987), we should expect CARP polyhedra to be extremely complicated. And they are indeed.

To make matters worse, it turns out that there are a large number of competing formulations in the literature. We will attempt to explain the motivation behind each of these in the present subsection.

In real-life CARP instances, it is common for the upper bound k on the number of vehicles to be small. Moreover, real problems are frequently defined on road networks, with the result that G is very sparse (most vertices have degree smaller than 5). Under such circumstances it is 'natural' to use $\mathcal{O}(k.|E|)$ variables ($\mathcal{O}(|E|)$ for each vehicle). This is the approach taken by Belenguer (1990), Welz (1994) and Belenguer & Benavent (1998a). We will call these 'sparse' formulations.

When k is large, or when $|E_R|$ is small relative to $|E|$, an alternative approach presents itself. We can 'break' the graph G apart as follows: a complete graph G' is constructed with two vertices for each required edge, representing the two endpoints, together with an extra vertex representing the depot. An edge >from one vertex to another in this expanded graph represents a shortest path between the corresponding pair of vertices in G. This leads to what we will call the 'dense' formulation, with $2|E_R|^2$ variables. This approach was explored by Letchford (1997a).

A third approach, suggested independently by Letchford (1997a) and Belenguer & Benavent (1998b), is to have only $|E|$ variables, one for each edge. Each variable represents the number of times a particular edge is traversed without being serviced. This leads to what we call a 'supersparse' formulation. Such a formulation is highly economical, elegant and easy to understand. However, it comes at a price: the individual vehicle routes are effectively 'tangled up', in that a feasible solution to such a formulation gives no indication of which vehicle traverses which edge. In fact, the problem of 'untangling' the routes appears to be $\mathcal{NP}$-hard, since it contains the $\mathcal{NP}$-hard *Bin Packing Problem* (see Garey & Johnson, 1979) as a special case. Nevertheless, the supersparse formulation is extremely valuable for producing *lower bounds* on the cost of a feasible solution.

Since sparse formulations have been given most attention in the literature, we devote Subsection 6.2 to them and go into considerable detail. Subsection 6.3 reviews the results on the dense and supersparse formulations.

Still other approaches to the CARP have been proposed. Golden & Wong (1981) gave a formulation in which there were an *exponential* number of variables and constraints. It is not worth examining this, however, since it was shown in the thesis of Welz (1994) that the lower bound obtained from the LP relaxation of this formulation is always zero. Finally, one could also have a formulation in which there is a variable for each feasible vehicle route and a constraint for each required edge ensuring that the edge is serviced. This approach is not examined here as it is the subject of Chapter 8.

In the remainder of this section we will use the convention that the depot is vertex 1.

6.2. SPARSE FORMULATIONS OF THE CARP

Two different sparse CARP formulations have appeared in the literature. One can be found in Welz (1994), the other in Belenguer (1990) and Belenguer & Benavent (1998a).

The formulation of Welz bears some similarities to the Golden & Wong (1981) formulation mentioned in the previous subsection, the crucial difference being that it has a polynomial number of variables. In this formulation, the problem is effectively converted into a directed problem. That is, each edge $\{i, j\}$ is regarded as two arcs (i, j) and (j, i), with identical costs. Then, if $\{i, j\}$ is required, we require that exactly one of the pair (i, j) and (j, i) is serviced. An advantage of viewing the CARP in this way is that one obtains a pure 0-1 formulation: it is easy to show that it is never necessary for any vehicle to traverse an edge more than once in a given direction.

The variables are defined as follows:

Let x_{ij}^p take the value 1 if arc (i, j) is traversed by vehicle p, 0 otherwise.

Let l_{ij}^p take the value 1 if arc (i,j) is serviced by vehicle p, 0 otherwise.

Let A be the set of arcs in the resulting directed graph (that is, $|A| = 2|E|$) and let $A(S)$ be the set of arcs with both end-vertices in S. Welz suggests defining the set $\mathcal{S}$ of feasible solutions by the following system:

$$x^p(\delta^+(i)) = x^p(\delta^-(i))(\forall i \in V, p = 1,\dots,k) \tag{6.43}$$

$$\sum_{p=1}^{k}(l_{ij}^p + l_{ji}^p) = 1(\forall \{i,j\} \in E_R) \tag{6.44}$$

$$x_{ij}^p \geq l_{ij}^p(\forall (i,j), p = 1,\dots,k) \tag{6.45}$$

$$\sum_{(i,j)\in A} q_{ij} l_{ij}^p \leq Q(\forall p = 1,\dots,k) \tag{6.46}$$

$$x^p(\delta^+(S)) \geq \frac{x^p(A(S))}{|A(S)|}(\forall S \subseteq V \setminus \{1\}) \tag{6.47}$$

$$x \in \{0,1\}^{2k|E|}, l \in \{0,1\}^{2k|E_R|} \tag{6.48}$$

The equations (6.43) ensure that each vehicle departs from each vertex as many times as it enters. The equations (6.44) ensure that each required edge is serviced exactly once. The inequalities (6.45) ensure that each vehicle traverses each edge that it services, (6.46) impose the capacity restrictions and (6.47) are connectivity inequalities. To tighten this basic formulation Welz proposes the following *odd-cut* inequalities:

$$\sum_{p=1}^{k} x^p(\delta^+(S)) \geq \lceil |\delta_R(S)|/2 \rceil \quad (\forall S \subseteq V \setminus \{1\} : |\delta_R(S)| \text{ odd}). \tag{6.49}$$

Also, Welz mentions that, if it is known that all k vehicles must be used, then the inequalities

$$x^p(\delta^+(1) \geq 1 \quad (\forall p = 1,\dots,k) \tag{6.50}$$

are valid also.

We would like to mention that the *connectivity* inequalities (6.47) proposed by Welz are in fact very weak. They can easily be disaggregated to give:

$$x^p(\delta^+(S)) \geq x_{ij}^p \quad (\forall S \subseteq V \setminus \{1\}, (i,j) \in E(S)).$$

We now move on to the sparse CARP formulation presented in Belenguer (1990) and Belenguer & Benavent (1998a), which uses less variables than the Welz formulation, is more 'natural' and gives better computational results. For each $e \in E_R$ and each $p = 1,\dots,k$, let x_e^p take the value 1 if e is serviced by vehicle k, 0 otherwise. Also, for each $e \in E$

and each $p = 1, \ldots, k$, define a general integer variable y_e^p representing the number of times e is traversed (*without* being serviced, if $e \in E_R$). The set $\mathcal{S}$ of feasible solutions is then given by:

$$x^p(\delta(i)) + y^p(\delta(i)) \equiv 0 \pmod 2 (\forall i \in V, p = 1, \ldots, k) \tag{6.51}$$

$$\sum_{p=1}^{k} x_e^p = 1 (\forall e \in E_R) \tag{6.52}$$

$$\sum_{e \in E_R} q_e x_e^p \leq Q (\forall p = 1, \ldots, k) \tag{6.53}$$

$$x^p(\delta(S)) + y^p(\delta(S)) \geq 2x_e^p (\forall S \subseteq V \setminus \{1\}, e \in E_R(S)) \tag{6.54}$$

$$x \in \{0,1\}^{k|E_R|}, y \in Z^{k|E|} \tag{6.55}$$

The reader who has persevered this far will have little difficulty interpreting the constraints in this formulation. The following results are given by Belenguer & Benavent for the associated P_I:

- The constraints (6.52) and (6.53), together with the binary conditions on the x variables in (6.55), define a so-called *Generalized Assignment polytope* (see Gottlieb & Rao, 1990a, b). Any inequality inducing a facet of this polytope (such as non-negativity inequalities $x_e^p \geq 0$ for all $e \in E_R$ and all $p = 1, \ldots, k$), induces a facet of P_I. Also, any implicit equation for this polytope (such as (6.52)) is an implicit equation for P_I and vice-versa.

- Computing the dimension of a Generalised Assignment polytope is $\mathcal{NP}$-hard, and therefore the same is true for the CARP polyhedron P_I.

- Non-negativity inequalities $y_e^p \geq 0$ for all $e \in E$ induce facets.

- If $S \subseteq V \setminus \{1\}$ is such that $|\delta_R(S)|$ is odd, then the *odd-cut* inequality

$$\sum_{p=1}^{k} y^p(\delta(S)) \geq 1 \tag{6.56}$$

  is valid and facet-inducing under mild conditions.

- If $S \subseteq V \setminus \{1\}$ is such that $\delta_R(S) \neq \emptyset$ and $F \subseteq \delta_R(S)$ is such that $|F|$ is odd, then the *parity* inequality

$$x^p(\delta_R(S) \setminus F) + y^p(\delta(S)) \geq x^p(F) - |F| + 1 \tag{6.57}$$

  is valid for $p = 1, \ldots k$ and facet-inducing under mild conditions.

- For a given $S \subset V \setminus \{1\}$, let $K(S)$ denote the minimum number of vehicles required to service $E_R(S) \cup \delta_R(S)$, due to the capacity restrictions. Then the *capacity* inequality

$$\sum_{p=1}^{k} y^p(\delta(S)) \geq 2K(S) - |\delta_R(S)| \tag{6.58}$$

is also valid. It will frequently induce a facet when $1 < K(S) < k$. When $K(S) = 1$, it will be dominated by the connectivity inequalities (6.54). When $K(S) = k$, then the capacity and connectivity inequalities are dominated by the stronger *obligatory* inequalities

$$x^p(\delta_R(S)) + y^p(\delta(S)) \geq 2 \tag{6.59}$$

for $p = 1, \ldots k$.

- If $S \subset V \setminus \{1\}$ and $\sum_{e \in E_R(S) \cup \delta_R(S)} \alpha_e x_e^p \leq \beta$ is valid for all p due to the Generalised Assignment polytope, then the inequality

$$x^p(\delta_R(S)) + y^p(\delta(S)) \geq \frac{2}{\beta}\Big(\sum_{e \in E_R(S) \cup \delta_R(S)} \alpha_e x_e^p\Big) \tag{6.60}$$

is valid also. Note that the connectivity inequalities (6.54) are a special case of this, since $x_e^p \leq 1$ is valid for the Generalised Assignment polytope.

In Letchford (1997a), some of these inequalities are generalised.

- Let $S \subseteq V \setminus \{1\}$ be such that $\delta_R(S) \neq \emptyset$, $F \subseteq \delta_R(S)$ be such that $|F|$ is odd and $H \subseteq \{1, \ldots k\}$ be an arbitrary (non-empty) set of vehicles. Then the *general parity* inequality

$$\sum_{p \in H} (x^p(\delta_R(S) \setminus F) + y^p(\delta(S))) \geq \sum_{p \in H} x^p(F) - |F| + 1 \tag{6.61}$$

is valid. It is easy to show that the general parity inequalities include odd-cut and parity inequalities as special cases.

- Let $S \subseteq V \setminus \{1\}$ be such that $K(S)$ vehicles are required to service $E_R(S) \cup \delta_R(S)$, due to the capacity restrictions. Let $H \subseteq \{1, \ldots k\}$ be an arbitrary subset of vehicles such that $k - K(S) < |H| \leq k$. Then the *minimum crossing* inequality

$$\sum_{p \in H} (x^p(\delta_R(S)) + y^p(\delta(S))) \geq 2(|H| - k + K(S)) \tag{6.62}$$

is valid. It is easy to show that the minimum crossing inequalities include capacity and obligatory inequalities as special cases.

The issue of when these inequalities induce facets is not examined by Letchford.

Finally, Letchford (1997a) also mentions that any valid inequality for the Corberán & Sanchis RPP formulation can easily be adapted to the Belenguer & Benavent CARP formulation. That is, if any inequality of the form $\sum_{e \in E} \alpha_e x_e \geq \beta$ is valid for the former, then $\sum_{p=1}^{k} \sum_{e \in E} \alpha_e y_{ep} \geq \beta$ is valid for the latter. The odd-cut inequalities come under this category. In general, however, the resulting inequalities are unlikely to induce facets unless the demands of the required edges are small relative to the vehicle capacity Q.

6.3. THE DENSE AND SUPERSPARSE FORMULATIONS OF THE CARP

In this subsection, the dense and supersparse approaches to the CARP are reviewed. We begin with the dense formulation, which was explored by Letchford (1997a).

Assume that the required edges are numbered from 1 to $|E_R|$. Define a complete graph $G'(V', E')$, with $1+2|R|$ vertices, as follows. The depot is represented by vertex 1 in G', just as it was in G. For $i = 1, \ldots, |E_R|$, vertex $i+1$ in V' represents one arbitrary end-vertex of required edge i. Similarly, vertex $i + |E_R| + 1$ in V' represents the other end-vertex of required edge i. Note that a single vertex in V may have multiple representatives in V'.

In E', for $i = 1, \ldots, |E_R|$, the edge $\{i+1, i+|E_R|+1\}$ now represents required edge i. The other edges in E' represent shortest paths between the corresponding vertices in G. We will let E^* denote these other edges. It can be readily checked that $|E^*| = 2|E_R|^2$. A $\{0,1\}$ variable x_{ij} is now defined for every edge in E^*, taking the value 1 if a vehicle traverses between i and j, 0 otherwise.

Now let $S \subseteq V' \setminus \{1\}$ be called *unbroken* if it has the property that, for $i = 2, \ldots, |E_R| + 1$, $i \in S$ if and only if $i + |E_R| \in S$. That is, S corresponds to a set of required edges in E_R. The set $\mathcal{S}$ of feasible solutions is then given by the following linear system (when $K(S)$ is defined as in the previous subsection):

$$x(\delta(i)) = 1 (i = 2, \ldots, 2|E_R| + 1) \quad (6.63)$$

$$x(\delta(S)) \geq 2K(S) (\forall S \subseteq V' \setminus \{1\}, S \text{ unbroken}) \quad (6.64)$$

$$x \in \{0,1\}^{|E^*|} \quad (6.65)$$

Letchford (1997a) first establishes a mapping between feasible solutions for this formulation and feasible solutions to a classical formulation for the well-known *Vehicle Routing Problem* (VRP). This means that valid inequalities for the latter formulation, such as *comb*, *multistar* and *hypotour* inequalities, can be 'borrowed' to give new inequalities for the CARP (for polyhedral results on the VRP, see, e.g., Araque, 1990; Araque, Hall & Magnanti, 1990; Cornuéjols & Harche, 1993; Augerat et al., 1995).

Another class of valid inequalities are presented by Letchford for the dense formulation. A set $S \subset V' \setminus \{1\}$ is called *broken* if it is not unbroken. If S is broken, then some set $F \neq \emptyset$ of required edges lies within $\delta(S)$ in G'. If F is odd, then the *blossom* inequality $x(\delta(S)) \geq 1$ is valid.

Now define the *enlargement* of a broken set S, denoted by $en(S)$, to be the minimum unbroken set $S' \subset V'$ such that $S \subset S'$. Letchford shows that a necessary condition for a blossom inequality to induce a facet is that $|F| \geq 2K(en(S)) + 1$, since, otherwise, it is dominated by a capacity inequality (6.64).

We now move on to examine the supersparse approach, which was proposed independently by Letchford (1997a) and Belenguer & Benavent (1998b). Let the general integer variable x_e represent the number of times e is traversed without being serviced. A feasible solution then represents a collection of superimposed routes.

At this point the reader may realise that it is far from obvious how to define the set $\mathcal{S}$ of feasible solutions in terms of equations, inequalities or congruences. Perhaps surprisingly, however, that does not stop us from producing valid inequalities. For example, Letchford (1997a) proposes the following inequalities

- RPP-type inequalities. Any inequality valid for the Corberán & Sanchis RPP formulation is valid for the supersparse CARP formulation. This includes non-negativity inequalities $x_e \geq 0$ for all $e \in E$ and R-odd cut inequalities $x(\delta(S)) \geq 1$ for all $S \subset V$ with $|\delta_R(S)|$ odd.
- Capacity inequalities. As usual, let $K(S)$ for any $S \subset V \setminus \{1\}$ denote the minimum number of vehicles required to service $E_R(S) \cup$

$\delta_R(S)$. Then the inequality

$$x(\delta(S)) \geq 2K(S) - |\delta_R(S)| \tag{6.66}$$

is valid.

Belenguer & Benavent (1998b) propose some further inequalities for the supersparse formulation. For any $S \subset V$, let $\alpha(S)$ be a lower bound on the minimum number of times that $\delta(S)$ must be traversed without servicing. If $|\delta_R(S)|$ is even, a natural value of $\alpha(S)$ is $\max\{0, 2K(S) - |\delta_R(S)|\}$. If $|\delta_R(S)|$ is odd, a natural value is $\max\{1, 2K(S) - |\delta_R(S)|\}$. Now suppose that $S_1, \ldots S_r$ are distinct subsets of $V \setminus \{1\}$ and that $F \subset E$ is such that there is no feasible solution to the CARP in which $x(\delta(S_i)) = \alpha(S_i)$ for all i, yet $x_e = 0$ for all $e \in F$. Then the following inequality is valid:

$$\sum_{i=1}^{r} x(\delta(S_i)) + 2x(F) \geq \sum_{i=1}^{r} \alpha(S_i) + 2. \tag{6.67}$$

These inequalities are related to the *hypotour* inequalities for the STSP (e.g., Grötschel & Padberg, 1985) and the *extended hypotour* inequalities for the VRP (e.g., Augerat et al., 1995). Belenguer & Benavent (1998b) give a heuristic for identifying suitable families of sets S_i, and then show how to find an appropriate set $F \subset E$ by solving a series of minimum cost flow problems.

7. CONCLUSIONS

In this chapter we have reviewed the known polyhedral results for a number of fundamental Arc Routing Problems. It will be seen that a great deal has been learned. Nevertheless, the results in the field of Arc Routing are not as comprehensive as the results known for certain node-routing problems, especially the Symmetric and Asymmetric Travelling Salesman Problems (see Jünger, Reinelt & Rinaldi, 1995, 1997).

Of course, problems which are encountered in practice are often more complex than the problems outlined here. Only recently have researchers begun to investigate the polyhedral structure of problems with more realistic constraints. To close this chapter, we mention a paper of our own which deals with a real-life problem.

Letchford & Eglese (1997) define a variant of the RPP called the *Rural Postman Problem with Deadline Classes*, in which the edges requiring service are partitioned into a small number of classes in order of priority. The idea here is that some roads might need to be treated within two hours, some within four hours, etc. This occurs in a number of practical applications, such as postal delivery, snow ploughing or winter gritting. Letchford & Eglese give a formulation in which the route is divided into 'time phases', each with their own set of variables. The resulting polyhedron is extremely complex, yet the theoretical results which were obtained were sufficient to yield a reasonable optimization algorithm (see Chapter 7 for more details).

The authors would like to encourage other researchers to examine more complex Arc Routing Problems from a polyhedral viewpoint.

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References

[1] J.R. Araque (1990) Lots of combs of different sizes for vehicle routing. *Discussion Paper*, Centre for Operations Research and Econometrics, Catholic University of Louvain, Belgium.

[2] J.R. Araque, L.A. Hall & T.L. Magnanti (1990) Capacitated trees, capacitated routing and associated polyhedra. *Discussion Paper*, Centre for Operations Research and Econometrics, Catholic University of Louvain, Belgium.

[3] A.A. Assad & B.L. Golden (1995) Arc routing methods and applications. In M.O. Ball, T.L. Magnanti, C.L. Monma & G.L. Nemhauser (Eds.) *Network Routing*. Handbooks of Operations Research and Management Science, 8. Amsterdam: North Holland.

[4] P. Augerat, J.M. Belenguer, E. Benavent, A. Corber·n, D. Naddef & G. Rinaldi (1995). Computational results with a branch-and-cut code for the capacitated vehicle routing problem. Research report RR949-M, ARTEMIS-IMAG, France.

[5] M.O. Ball & M.J. Magazine (1988) Sequencing of insertions in printed circuit board assembly. *Oper. Res.*, *36*, 192-201.

[6] J.M. Belenguer (1990) The capacitated arc-routing problem polyhedron. *PhD dissertation* (in Spanish), Dept. of Stats and OR, University of Valencia, Spain.

[7] J.M. Belenguer & E. Benavent (1998a) The capacitated arc routing problem: valid inequalities and facets. *Computational Optimization & Applications*, *10*, 165-187.

[8] J.M. Belenguer & E. Benavent (1998b) A cutting-plane algorithm for the capacitated arc routing problem. *Working Paper*, Dept. of Stats and OR, University of Valencia, Spain.

[9] T. Christof & G. Reinelt (1996) Combinatorial optimization and small polytopes. *Top (J. Span. Stats. & O.R. Soc.)*, *4*, 1-64.

[10] N.Christofides (1973) The optimum traversal of a graph. *Omega*, *1*, 719-732.

[11] N.Christofides, E. Benavent, V. Campos, A. Corberán & E. Mota (1984) An optimal method for the mixed postman problem, in P. Thoft-Christensen (Ed.) *System Modelling and Optimization*, Lecture Notes in Control and Inf. Sciences, 59. Berlin: Springer.

[12] N. Christofides, V. Campos, A. Corberán & E. Mota (1981) An algorithm for the rural postman problem. *Report IC.O.R.81.5*, Imperial College, London.

[13] N. Christofides, V. Campos, A. Corberán & E. Mota (1986) An algorithm for the rural postman problem on a directed graph. *Math. Prog. Study*, *26*, 155-166.

[14] A. Corberán, A. Romero & J.M. Sanchis (1997) Facets of the mixed rural postman polyhedron. *Working paper*, Dept. of Stats and OR, University of Valencia, Spain.

[15] A. Corberán & J.M. Sanchis (1994) A polyhedral approach to the rural postman problem. *Eur. J. Opl Res.*, *79*, 95-114.

[16] A. Corberán & J.M. Sanchis (1998) The general routing problem polyhedron: facets from the RPP and GTSP polyhedra. *Eur. J. Opl Res.*, *108*, 538-550.

[17] G. Cornuéjols & F. Harche (1993) Polyhedral study of the capacitated vehicle routing problem. *Math. Prog.*, *60*, 21-52.

[18] G. Cornuéjols, J. Fonlupt & D. Naddef (1985) The travelling salesman problem on a graph and some related integer polyhedra. *Math. Prog.*, *33*, 1-27.

[19] J. Edmonds (1963)The Chinese postman problem. *Oper. Res.*, *13*, Suppl. 1, B73-B77.

[20] J. Edmonds & E.L. Johnson (1973) Matchings, Euler tours and the Chinese postman. *Math. Prog.*, *5*, 88-124.

[21] H.A. Eiselt, M. Gendreau & G. Laporte (1995) Arc-routing problems, part 1: the Chinese postman problem. *Oper. Res.*, *43*, 231-242.

[22] H.A. Eiselt, M. Gendreau & G. Laporte (1995) Arc-routing problems, part 2: the rural postman problem. *Oper. Res.*, *43*, 399-414.

[23] B. Fleischmann (1985) A cutting-plane procedure for the travelling salesman problem on a road network. *Eur. J. Opl Res.*, *21*, 307-317.

[24] L.R. Ford & D.R. Fulkerson (1962) *Flows in Networks*. Princeton University Press, Princeton, NJ.

[25] M.R. Garey & D.S. Johnson (1979) *Computers and Intractability: a guide to the theory of NP-completeness*. San Fr.; Freeman.

[26] G. Ghiani & G. Laporte (1997) A branch-and-cut algorithm for the undirected rural postman problem. *Working Paper*, Centre for Research on Transportation, University of Montréal.

[27] B.L. Golden & R.T. Wong (1981) Capacitated arc routing problems. *Networks*, *11*, 305-315.

[28] E.S. Gottlieb & M.R. Rao (1990a) The generalised assignment problem: valid inequalities and facets. *Math. Program.*, *46*, 31-52

[29] E.S. Gottlieb & M.R. Rao (1990b) (1, k)-configuration facets for the generalised assignment problem. *Math. Program.*, *46*, 53-60.

[30] M. Grötschel & M.W. Padberg (1985) Polyhedral theory. In E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy-Kan & D.B. Schmoys (eds.) *The Travelling Salesman Problem*. New York: Wiley.

[31] M. Grötschel & Z. Win (1988) On the windy postman polyhedron. Report No. 75, Schwerpunkt-progam der Deutschen Forschungsgeneinschaft, Universität Augsburg, Germany.

[32] M. Grötschel & Z. Win (1992) A cutting plane algorithm for the windy postman problem. Math. Prog., *55*, 339-358.

[33] M. Guan (1962) Graphic programming using odd or even points. *Chinese Math.*, *1*, 237-277.

[34] M. Guan (1984) On the windy postman problem. *Discr. Appl. Math.*, *9*, 41-46.

[35] H. Gun (1993) Polyhedral structure and efficient algorithms for certain classes of directed rural postman problem. *PhD dissertation*, Applied Math. Program, University of Maryland at College Park, Md.

[36] M. Jünger, G. Reinelt & G. Rinaldi (1995) The travelling salesman problem. In M.O. Ball, T.L. Magnanti, C.L. Monma & G.L. Nemhauser (Eds.) *Network Models*. Handbooks on Operations Research and Management Science Vol. 7. Amsterdam: Elsevier.

[37] M. Jünger, G. Reinelt & G. Rinaldi (1997) The travelling salesman problem. In M. Dell'Amico, F. Maffioli & S. Martello (Eds.) *Annotated Bibliographies in Combinatorial Optimization*. Chichester: Wiley.

[38] C.H. Kappauf & G.J. Koehler (1979) The mixed postman problem. *Disc. Appl. Math.*, 1, 89-103.

[39] R.M. Karp & C.H. Papadimitriou (1982) On linear characterizations of combinatorial optimization problems. *SIAM J. Comput.*, *11*, 620-632.

[40] J.K. Lenstra & A.H.G. Rinnooy-Kan (1976) On general routing problems. *Networks*, *6*, 273-280.

[41] A.N. Letchford (1997a) Polyhedral results for some constrained arc-routing problems. *PhD Dissertation*, Dept. of Management Science, Lancaster University.

[42] A.N. Letchford (1997b) New inequalities for the general routing problem. *Eur. J. Opl Res.*, *96*, 317-322.

[43] A.N. Letchford (1997c) The general routing polyhedron: a unifying framework. To appear in *Eur. J. Opl Res.*

[44] A.N. Letchford & R.W. Eglese (1998) The rural postman problem with deadline classes. *Eur. J. of Opl Res.*, *105*, 390-400.

[45] G.L. Nemhauser & L.A. Wolsey (1988) *Integer and Combinatorial Optimization*. New York: Wiley.

[46] Y. Nobert & J.-C. Picard (1996) An optimal algorithm for the mixed Chinese postman problem. *Networks*, *27*, 95-108.

[47] C.S. Orloff (1974) A fundamental problem in vehicle routing. *Networks*, *4*, 35-64.

[48] M.W. Padberg & G. Rinaldi (1991) A branch-and-cut algorithm for the resolution of large-scale symmetric travelling salesman problems. *SIAM Rev.*, *33*, 60-100.

[49] C.H. Papadimitriou (1976) On the complexity of edge traversing. *J. of the A.C.M.*, *23*, 544-554.

[50] M. Queyranne & Y. Wang (1992) On the convex hull of feasible solutions to certain combinatorial problems. *Oper. Res. Lett.*, *11*, 1-11.

[51] T.K. Ralphs (1993) On the mixed Chinese postman problem. *Oper. Res. Lett.*, *14*, 123-127.

[52] A. Romero (1997) The rural postman polyhedron on a mixed graph. *PhD dissertation* (in Spanish), Dept. of Stats and OR, University of Valencia, Spain.

[53] J.V. Savall (1990) Polyhedral results and approximate algorithms for the directed rural postman problem. *PhD Dissertation* (in Spanish), Dept. of Stats and OR, University of Valencia, Spain.

[54] S.A. Welz (1994) Optimal solutions for the capacitated arc routing problem using integer programming. *PhD Dissertation*, Dept. of QT and OM, University of Cincinnati.

[55] H. Weyl (1935) Elementare Theorie der konvexen Polyeder. *Comentarii Mathematici Helvetica*, *7*, 290-306.

[56] Z. Win (1987) Contributions to routing problems. *Doctoral Dissertation*, Universität Augsburg, Germany.

[57] Z. Win (1989) On the windy postman problem on Eulerian graphs. *Math. Prog.*, *44*, 97-112.