Polynomial-Time Separation of Simple Comb Inequalities

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Abstract. The *comb* inequalities are a well-known class of facet-inducing inequalities for the Traveling Salesman Problem, defined in terms of certain vertex sets called the *handle* and the *teeth*. We say that a comb inequality is *simple* if the following holds for each tooth: either the intersection of the tooth with the handle has cardinality one, or the part of the tooth outside the handle has cardinality one, or both. The simple comb inequalities generalize the classical *2-matching* inequalities of Edmonds, and also the so-called *Chvátal comb* inequalities.

In 1982, Padberg and Rao [29] gave a polynomial-time algorithm for *separating* the 2-matching inequalities – i.e., for testing if a given fractional solution to an LP relaxation violates a 2-matching inequality. We extend this significantly by giving a polynomial-time algorithm for separating the simple comb inequalities. The key is a result due to Caprara and Fischetti.

1 Introduction

The famous *Symmetric Traveling Salesman Problem* (STSP) is the $\mathcal{NP}$-hard problem of finding a minimum cost Hamiltonian cycle (or *tour*) in a complete undirected graph. The most successful optimization algorithms at present (e.g., Padberg & Rinaldi [31], Applegate, Bixby, Chvátal & Cook [1]), are based on an integer programming formulation of the STSP due to Dantzig, Fulkerson & Johnson [7], which we now describe.

Let G be a complete graph with vertex set V and edge set E. For each edge $e \in E$, let c_e be the cost of traversing edge e. For any $S \subset V$, let $\delta(S)$ (respectively, $E(S)$), denote the set of edges in G with exactly one end-vertex (respectively, both end vertices) in S. Then, for each $e \in E$, define the 0-1 variable x_e taking the value 1 if e is to be in the tour, 0 otherwise. Finally let $x(F)$ for any $F \subset E$ denote $\sum_{e \in F} x_e$. Then the formulation is:

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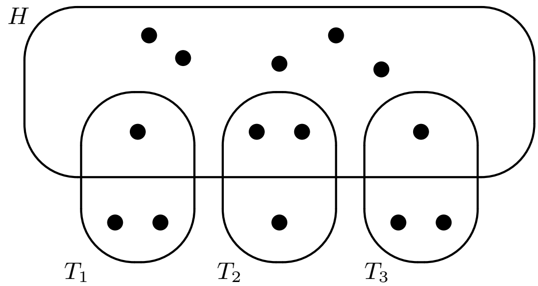

Fig. 1. A comb with three teeth

Minimize $\sum_{e \in E} c_e x_e$

Subject to:

$$x(\delta(\{i\})) = 2 \quad \forall i \in V, \tag{1}$$

$$x(E(S)) \leq |S| - 1 \quad \forall S \subset V : 2 \leq |S| \leq |V| - 2, \tag{2}$$

$$x_e \geq 0 \quad \forall e \in E, \tag{3}$$

$$x \in Z^{|E|} \ . \tag{4}$$

Equations (1) are called *degree equations*. The inequalities (2) are called *subtour elimination constraints* (SECs) and the inequalities (3) are simple non-negativity conditions. Note that an SEC with $|S| = 2$ is a mere upper bound of the form $x_e \leq 1$ for some edge e.

The convex hull in $\mathbb{R}^{|E|}$ of vectors satisfying (1) - (4) is called a *Symmetric Traveling Salesman Polytope*. The polytope defined by (1) - (3) is called a *Subtour Elimination Polytope*. These polytopes are denoted by STSP(n) and SEP(n) respectively, where $n := |V|$. Clearly, STSP(n) $\subseteq$ SEP(n), and containment is strict for $n \geq 6$.

The polytopes STSP(n) have been studied in great depth and many classes of valid and facet-inducing inequalities are known; see the surveys by Jünger, Reinelt & Rinaldi [19,20] and Naddef [22]. Here we are primarily interested in the *comb* inequalities of Grötschel & Padberg [15,16], which are defined as follows. Let $t \geq 3$ be an odd integer. Let $H \subset V$ and $T_j \subset V$ for $j = 1, \ldots, t$ be such that $T_j \cap H \neq \emptyset$ and $T_j \setminus H \neq \emptyset$ for $j = 1, \ldots, t$, and also let the T_j be vertex-disjoint. (See Fig. 1 for an illustration.) The comb inequality is:

$$x(E(H)) + \sum_{j=1}^{t} x(E(T_j)) \leq |H| + \sum_{j=1}^{t} |T_j| - \lceil 3t/2 \rceil \ . \tag{5}$$

The set H is called the *handle* of the comb and the T_j are called *teeth*.

Comb inequalities induce facets of STSP(n) for $n \geq 6$ [15,16]. The validity of comb inequalities in the special case where $|T_j \cap H| = 1$ for all j was proved by Chvátal [6]. For this reason inequalities of this type are sometimes referred to as *Chvátal comb* inequalities. If, in addition, $|T_j \setminus H| = 1$ for all j, then the inequalities reduce to the classical *2-matching* inequalities of Edmonds [8].

In this paper we are concerned with a class of inequalities which is intermediate in generality between the class of comb inequalities and the class of Chvátal comb inequalities. For want of a better term, we call them *simple* comb inequalities, although the reader should be aware that the term *simple* is used with a different meaning in Padberg & Rinaldi [30], and with yet another meaning in Naddef & Rinaldi [23,24].

Definition 1. *A comb (and its associated comb inequality) will be said to be* simple *if, for all* j*, either* $|T_j \cap H| = 1$ *or* $|T_j \setminus H| = 1$ *(or both).*

So, for example, the comb shown in Fig. 1 is simple because $|T_1 \cap H|$, $|T_2 \setminus H|$ and $|T_3 \cap H|$ are all equal to 1. Note however that it is not a Chvátal comb, because $|T_2 \cap H| = 2$.

For a given class of inequalities, a *separation algorithm* is a procedure which, given a vector $x^* \in \mathbb{R}^{|E|}$ as input, either finds an inequality in the class which is violated by x^*, or proves that none exists (see Grötschel, Lovász & Schrijver [14]). A desirable property of a separation algorithm is that it runs in polynomial time.

In 1982, Padberg & Rao [29] discovered a polynomial-time separation algorithm for the *2-matching* inequalities. First, they showed that the separation problem is equivalent to the problem of finding a *minimum weight odd cut* in a certain weighted labeled graph. This graph has $\mathcal{O}(|E^*|)$ vertices and edges, where $E^* := \{e \in E : x^*_e > 0\}$. Then, they proved that the desired cut can be found by solving a sequence of $O(|E^*|)$ max-flow problems. Using the well-known *pre-flow push* algorithm (Goldberg & Tarjan [12]) to solve the max-flow problems, along with some implementation tricks given in Grötschel & Holland [13], the Padberg-Rao separation algorithm can be implemented to run in $\mathcal{O}(n|E^*|^2 \log(n^2/|E^*|))$ time, which is $\mathcal{O}(n^5)$ in the worst case, but $\mathcal{O}(n^3 \log n)$ if the support graph is sparse. (The *support graph*, denoted by G^*, is the subgraph of G induced by E^*.)

In Padberg & Grötschel [28], page 341, it is conjectured that there also exists a polynomial-time separation algorithm for the more general *comb* inequalities. This conjecture is still unsettled, and in practice many researchers resort to *heuristics* for comb separation (see for example Padberg & Rinaldi [30], Applegate, Bixby, Chvátal & Cook [1], Naddef & Thienel [25]). Nevertheless, some progress has recently been made on the theoretical side. In chronological order:

- Carr [5] showed that, for a *fixed* value of t, the comb inequalities with t teeth can be separated by solving $\mathcal{O}(n^{2t})$ maximum flow problems, i.e., in $\mathcal{O}(n^{2t+1}|E^*| \log(n^2/|E^*|))$ time using the pre-flow push algorithm.
- Fleischer & Tardos [9] gave an $\mathcal{O}(n^2 \log n)$ algorithm for detecting *maximally violated* comb inequalities. (A comb inequality is maximally violated if it is violated by $\frac{1}{2}$, which is the largest violation possible if $x^* \in SEP(n)$.) However this algorithm only works when G^* is planar.
- Caprara, Fischetti & Letchford [3] showed that the comb inequalities were contained in a more general class of inequalities, called $\{0, \frac{1}{2}\}$-*cuts*, and showed how to detect maximally violated $\{0, \frac{1}{2}\}$-cuts in $\mathcal{O}(n^2|E^*|)$ time.

- Letchford [21] defined a different generalization of the comb inequalities, called *domino-parity* inequalities, and showed that the associated separation problem can be solved in $\mathcal{O}(n^3)$ time when G^* is planar.
- Caprara & Letchford [4] showed that, if the *handle* H is fixed, then the separation problem for $\{0, \frac{1}{2}\}$-cuts can be solved in polynomial time. They did not analyze the running time, but the order of the polynomial is likely to be very high.

In this paper we make another step forward in this line of research, by proving the following theorem:

Theorem 1. *Simple comb inequalities can be separated in polynomial time, provided that* $x^* \in SEP(n)$.

This is a significant extension of the Padberg-Rao result. The proof is based on some results of Caprara & Fischetti [2] concerning $\{0, \frac{1}{2}\}$-cuts, together with an 'uncrossing' argument which enables one to restrict attention to a small (polynomial-sized) collection of *candidate teeth*.

The structure of the paper is as follows. In Sect. 2 we summarize the results given in [2] about $\{0, \frac{1}{2}\}$-cuts and show how they relate to the simple comb inequalities. In Sect. 3 the uncrossing argument is given. In Sect. 4 we describe the separation algorithm and analyze its running time, which turns out to be very high at $\mathcal{O}(n^9 \log n)$. In Sect. 5, we show that the running time can be reduced to $\mathcal{O}(n^3|E^*|^3 \log n)$, and suggest ways in which it could be reduced further. Conclusions are given in Sect. 6.

2 Simple Comb Inequalities as $\{0, \frac{1}{2}\}$-Cuts

As mentioned above, we will need some definitions and results from Caprara & Fischetti [2]. We begin with the definition of $\{0, \frac{1}{2}\}$-*cuts*:

Definition 2. *Given an integer polyhedron* $P_I := conv\{x \in Z_+^q : Ax \leq b\}$, *where* A *is a* $p \times q$ *integer matrix and* b *is a column vector with* p *integer entries, a* $\{0, \frac{1}{2}\}$-cut *is a valid inequality for* P_I *of the form*

$$\lfloor \lambda A \rfloor x \leq \lfloor \lambda b \rfloor, \tag{6}$$

where $\lambda \in \{0, \frac{1}{2}\}^p$ *is chosen so that* λb *is not integral.*

(Actually, Caprara & Fischetti gave a more general definition, applicable when variables are not necessarily required to be non-negative; but the definition given here will suffice for our purposes. Also note that an equation can easily be represented by two inequalities.)

Caprara and Fischetti showed that many important classes of valid and facet-inducing inequalities, for many combinatorial optimization problems, are $\{0, \frac{1}{2}\}$-cuts. They also showed that the associated separation problem is strongly $\mathcal{NP}$-hard in general, but polynomially solvable in certain special cases. To present these special cases, we need two more definitions:

Definition 3. *The* mod-2 support *of an integer matrix A is the matrix obtained by replacing each entry in A by its parity (0 if even, 1 if odd).*

Definition 4. *A $p \times q$ binary matrix A is called an* edge-path incidence matrix of a tree *(EPT for short), if there is a tree T with p edges such that each column of A is the characteristic vector of the edges of a path in T.*

The main theorem in [2] is then the following:

Theorem 2 (Caprara & Fischetti [2]). *The separation problem for $\{0, \frac{1}{2}\}$-cuts for a system $Ax \leq b$ can be solved in polynomial time if the mod-2 support of A, or its transpose, is EPT.*

Now let us say that the ith inequality in the system $Ax \leq b$ is *used* if its multiplier is non-zero, i.e., if $\lambda_i = \frac{1}{2}$. Moreover, we will also say that the non-negativity inequality for a given variable x_i has been *used* if the i-th coefficient of the vector λA is fractional. The reasoning behind this is that rounding down the coefficient of x_i on the left hand side of (6) is equivalent to adding one half of the non-negativity inequality $x_i \geq 0$ (written in the reversed form, $-x_i \leq 0$).

Given these definitions, it can be shown that:

Proposition 1 (Caprara & Fischetti [2]). *Let $x^* \in \mathbb{R}^q_+$ be a point to be separated. Then a $\{0, \frac{1}{2}\}$-cut is violated by x^* if and only if the sum of the slacks of the inequalities used, computed with respect to x^*, is less than* 1.

Under the (reasonable) assumption that $Ax^* \leq b$, all slacks are non-negative and Proposition 1 also implies that the slack of each inequality used must be less than 1.

The reason that these results are of relevance is that comb inequalities can be derived as $\{0, \frac{1}{2}\}$-cuts from the degree equations and SECs; see Caprara, Fischetti & Letchford [3] for details. However, we have not been able to derive a polynomial-time separation algorithm for comb inequalities (simple or not) based on this observation alone. Instead we have found it necessary to consider a certain *weakened version* of the SECs, presented in the following lemma:

Lemma 1. *For any $S \subset V$ such that $1 \leq |S| \leq |V| - 2$, and any $i \in V \setminus S$, the following* tooth *inequality is valid for STSP(n):*

$$2x(E(S)) + x(E(i : S)) \leq 2|S| - 1, \tag{7}$$

where $E(i : S)$ denotes the set of edges with i as one end-node and the other end-node in S.

Proof: The inequality is the sum of the SEC on S and the SEC on $S \cup \{i\}$.

We will call i the 'root' of the tooth and S the 'body'.

The next proposition shows that, if we are only interested in *simple* comb inequalities, we can work with the tooth inequalities instead of the (stronger) SECs:

Proposition 2. *Simple comb inequalities can be derived as* $\{0, \frac{1}{2}\}$*-cuts from the degree equations (1) and the tooth inequalities (7).*

Proof: First, sum together the degree equations for all $i \in H$ to obtain:

$$2x(E(H)) + x(\delta(H)) \leq 2|H|. \tag{8}$$

Now suppose, without loss of generality, that there is some $1 \leq k \leq t$ such that $|T_j \cap H| = 1$ for $j = 1, \ldots, k$, and $|T_j \setminus H| = 1$ for $k+1, \ldots, t$. For $j = 1, \ldots, k$, associate a tooth inequality of the form (7) with tooth T_j, by setting $\{i\} := T_j \cap H$ and $S := T_j \setminus H$. Similarly, for $j = k+1, \ldots, t$, associate a tooth inequality with tooth T_j, by setting $\{i\} := T_j \setminus H$ and $S := T_j \cap H$. Add all of these tooth inequalities to (8) to obtain:

$$2x(E(H)) + x(\delta(H)) + \sum_{j=1}^{k} (2x(E(T_j \setminus H)) + x(E(T_j \cap H : T_j \setminus H)))$$

$$+ \sum_{j=k+1}^{t} (2x(E(T_j \cap H)) + x(E(T_j \cap H : T_j \setminus H))) \leq 2|H| + 2\sum_{j=1}^{t} |T_j| - 3t.$$

This can be re-arranged to give:

$$2x(E(H)) + 2\sum_{j=1}^{t} x(E(T_j)) + x\left(\delta(H) \setminus \bigcup_{j=1}^{t} E(T_j \cap H : T_j \setminus H)\right)$$

$$\leq 2|H| + 2\sum_{j=1}^{t} |T_j| - 3t.$$

Dividing by two and rounding down yields (5).

Although not crucial to the remainder of this paper, it is interesting to note that the SECs (2) can themselves be regarded as 'trivial' $\{0, \frac{1}{2}\}$-cuts, obtained by dividing a single tooth inequality by two and rounding down. Indeed, as we will show in the full version of this paper, any $\{0, \frac{1}{2}\}$-cut which is derivable from the degree equations and tooth inequalities is either an SEC, or a simple comb inequality, or dominated by these inequalities.

3 Candidate Teeth: An Uncrossing Argument

Our goal in this paper is to apply the results of the previous section to yield a polynomial-time separation algorithm for simple comb inequalities. However, a problem which immediately presents itself is that there is an exponential number of tooth inequalities, and therefore the system $Ax \leq b$ defined by the degree and tooth inequalities is of exponential size.

Fortunately, Proposition 1 tells us that we can restrict our attention to tooth inequalities whose slack is less than 1, without losing any violated $\{0, \frac{1}{2}\}$-cuts. Such tooth inequalities are polynomial in number, as shown in the following two lemmas.

Lemma 2. *Suppose that* $x^* \in SEP(n)$*. Then the number of sets whose SECs have slack less than* $\frac{1}{2}$ *is* $\mathcal{O}(n^2)$*, and these sets can be found in* $\mathcal{O}(n|E^*|(|E^*| + n \log n))$ *time.*

Proof: The degree equations can be used to show that the slack of the SEC on a set S is less than $\frac{1}{2}$ if and only if $x^*(\delta(S)) < 3$. Since the minimum cut in G^* has weight 2, we require that the cut-set $\delta(S)$ has a weight strictly less than $\frac{3}{2}$ times the weight of the minimum cut. It is known (Hensinger & Williamson [18]) that there are $\mathcal{O}(n^2)$ such sets, and that the algorithm of Nagamochi, Nishimura & Ibaraki [26] finds them in $\mathcal{O}(n|E^*|(|E^*| + n \log n))$ time.

Lemma 3. *Suppose that* $x^* \in SEP(n)$*. Then the number of distinct tooth inequalities with slack less than* 1 *is* $\mathcal{O}(n^3)$*, and these teeth can be found in* $\mathcal{O}(n|E^*|(|E^*| + n \log n))$ *time.*

Proof: The slack of the tooth inequality is equal to the slack of the SEC for S plus the slack of the SEC for $\{i\} \cup S$. For the tooth inequality to have slack less than 1, the slack for at least one of these SECs must be less than $\frac{1}{2}$. So we can take each of the $\mathcal{O}(n^2)$ sets mentioned in Lemma 2 and consider them as candidates for either S or $\{i\} \cup S$. For each candidate, there are only n possibilities for the root i. The time bottleneck is easily seen to be the Nagamochi, Nishimura & Ibaraki algorithm.

Now consider the system of inequalities $Ax \leq b$ formed by the degree equations and the $\mathcal{O}(n^3)$ tooth inequalities mentioned in Lemma 3. If we could show that the mod-2 support of A (or its transpose) is always an EPT matrix, then we would be done. Unfortunately this is *not* the case. (It is easy to produce counter-examples even for $n = 6$.)

Therefore we must use a more involved argument if we wish to separate simple comb inequalities via $\{0, \frac{1}{2}\}$-cut arguments. It turns out that the key is to pay special attention to tooth inequalities whose slack is strictly less than $\frac{1}{2}$. This leads us to the following definition and lemma:

Definition 5. *A tooth in a comb is said to be* light *if the slack of the associated tooth inequality is less than* $\frac{1}{2}$*. Otherwise it is said to be* heavy.

Lemma 4. *If a simple comb inequality is violated by a given* $x^* \in SEP(n)$*, then at most one of its teeth can be heavy.*

Proof: If two of the teeth in a comb are heavy, the slacks of the associated tooth inequalities sum to at least $\frac{1}{2} + \frac{1}{2} = 1$. Then, by Proposition 1, the comb inequality is not violated.

We now present a slight refinement of the light/heavy distinction.

Definition 6. *For a given* $i \in V$*, a vertex set* $S \subset V \setminus \{i\}$ *is said to be* i-light *if the tooth inequality with root* i *and body* S *has slack strictly less than* $\frac{1}{2}$*. Otherwise it is said to be* i-heavy.

Now we invoke our uncrossing argument. The classical definition of *crossing* is as follows:

Definition 7. *Two vertex sets $S_1, S_2 \subset V$ are said to* cross *if each of the four sets $S_1 \cap S_2$, $S_1 \setminus S_2$, $S_2 \setminus S_1$ and $V \setminus (S_1 \cup S_2)$ is non-empty.*

However, we will need a slightly modified definition:

Definition 8. *Let $i \in V$ be a fixed root. Two vertex sets $S_1, S_2 \subset V \setminus \{i\}$ are said to i-*cross *if each of the four sets $S_1 \cap S_2$, $S_1 \setminus S_2$, $S_2 \setminus S_1$ and $V \setminus (S_1 \cup S_2 \cup \{i\})$ is non-empty.*

Theorem 3. *Let $i \in V$ be a fixed root. If $x^* \in SEP(n)$, it is impossible for two i-light sets to i-cross.*

Proof: If we sum together the degree equations (1) for all $i \in S_1 \cap S_2$, along with the SECs on the four vertex sets $i \cup S_1 \cup S_2$, $S_1 \setminus S_2$, $S_2 \setminus S_1$ and $S_1 \cap S_2$, then (after some re-arranging) we obtain the inequality:

$$x^*(E(i:S_1)) + 2x^*(E(S_1)) + x^*(E(i:S_2)) + 2x^*(E(S_2)) \leq 2|S_1| + 2|S_2| - 3. \quad (9)$$

On the other hand, the sum of the tooth inequality with root i and body S_1 and the tooth inequality with root i and body S_2 is:

$$x^*(E(i:S_1)) + 2x^*(E(S_1)) + x^*(E(i:S_2)) + 2x^*(E(S_2)) \leq 2|S_1| + 2|S_2| - 2. \quad (10)$$

Comparing (10) and (9) we see that the sum of the slacks of these two tooth inequalities is at least 1. Since $x^* \in SEP(n)$, each of the individual slacks is non-negative. Hence at least one of the slacks must be $\geq \frac{1}{2}$. That is, at least one of S_1 and S_2 is i-heavy.

Corollary 1. *For a given root i, there are only $\mathcal{O}(n)$ i-light vertex sets.*

Corollary 2. *The number of distinct tooth inequalities with slack less than $\frac{1}{2}$ is $\mathcal{O}(n^2)$.*

The following lemma shows that we can eliminate half of the i-light sets from consideration.

Lemma 5. *A tooth inequality with root i and body S is equivalent to the tooth inequality with root i and body $V \setminus (S \cup \{i\})$.*

Proof: The latter inequality can be obtained from the former by subtracting the degree equations for the vertices in S, and adding the degree equations for the vertices in $V \setminus (S \cup \{i\})$.

Therefore, for any i, we can pick any arbitrary vertex $j \neq i$ and eliminate all i-light sets containing j, without losing any violated $\{0, \frac{1}{2}\}$-cuts. It turns out that the remaining i-light sets have an interesting structure:

Lemma 6. *For any i and any arbitrary $j \neq i$, the i-light sets which do not contain j are* 'nested'*. That is, if S_1 and S_2 are i-light sets which do not contain j, then either S_1 and S_2 are disjoint, or one is entirely contained in the other.*

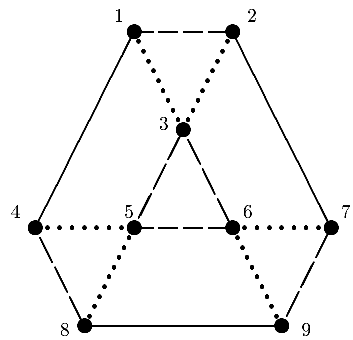

Fig. 2. A fractional point contained in SEP(9)

Proof: If S_1 and S_2 did not meet this condition, then they would i-cross.

We close this section with a small example.

Example: Fig. 2 shows the support graph G^* for a vector x^* which lies in SEP(9). The solid lines, dashed lines and dotted lines show edges with $x^*_e = 1$, 2/3 and 1/3, respectively. The 1-light sets are $\{2\}, \{4\}, \{3, \ldots, 9\}$ and $\{2, 3, 5, 6, 7, 8, 9\}$; the 3-light sets are $\{5\}, \{6\}, \{5, 6\}, \{1, 2, 4, 7, 8, 9\}, \{1, 2, 4, 5, 7, 8, 9\}$ and $\{1, 2, 4, 6, 7, 8, 9\}$. The 1-heavy sets include, for example, $\{3\}, \{4, 8\}$ and $\{3, 5, 6\}$. The 3-heavy sets include, for example, $\{1\}, \{2\}$ and $\{1, 4\}$. The reader can easily identify light and heavy sets for other roots by exploiting the symmetry of the fractional point.

4 Separation

Our separation algorithm has two stages. In the first stage, we search for a violated simple comb inequality in which all of the teeth are light. If this fails, then we proceed to the second stage, where we search for a violated simple comb inequality in which one of the teeth is heavy. Lemma 4 in the previous section shows that this approach is valid.

The following theorem is at the heart of the first stage of the separation algorithm:

Theorem 4. *Let $Ax \leq b$ be the inequality system formed by the degree equations (written in less-than-or-equal-to form), and, for each i, the tooth inequalities corresponding to the i-light sets forming a nested family obtained as in the previous section. Then the mod-2 support of the matrix A is an EPT matrix.*

The proof of this theorem is long and tedious and will be given in the full version of the paper. Instead, we demonstrate that the theorem is true for the fractional point shown in Fig. 2. It should be obvious how to proceed for other fractional points.

Example (Cont.): Consider once again the fractional point shown in Fig. 2. There are four i-light sets for each of the roots $1, 2, 4, 7, 8, 9$ and six i-light sets for the remaining three roots. Applying Lemma 5 we can eliminate half of these from consideration. So suppose we choose:

- 1-light sets: $\{2\}, \{4\}$;
- 2-light sets: $\{1\}, \{7\}$;
- 3-light sets: $\{5\}, \{6\}, \{5, 6\}$;
- 4-light sets: $\{1\}, \{8\}$;
- 5-light sets: $\{3\}, \{6\}, \{3, 6\}$;
- 6-light sets: $\{3\}, \{5\}, \{3, 5\}$;
- 7-light sets: $\{2\}, \{9\}$;
- 8-light sets: $\{4\}, \{9\}$;
- 9-light sets: $\{7\}, \{8\}$.

This leads to 21 light tooth inequalities in total. However, there are some duplicates: a tooth inequality with root i and body $\{j\}$ is identical to a tooth inequality with root j and body $\{i\}$ (in both cases the inequality is a simple upper bound, $x_{ij} \leq 1$). In fact there are only 12 distinct inequalities, namely:

$$2x_{35} + x_{36} + x_{56} \leq 3 \tag{11}$$

$$2x_{36} + x_{35} + x_{56} \leq 3 \tag{12}$$

$$2x_{56} + x_{35} + x_{36} \leq 3, \tag{13}$$

plus the upper bounds on $x_{12}, x_{14}, x_{27}, x_{35}, x_{36}$, x_{48}, x_{56}, x_{79} and x_{89}. Therefore the matrix A has 36 columns (one for each variable), and 21 rows (12 tooth inequalities plus 9 degree equations). In fact we can delete the columns associated with variables which are zero at x^*. This leaves only 15 columns; the resulting matrix A is as follows (the first three rows correspond to (11) - (13), the next nine to simple upper bounds, and the final nine to degree equations):

$$\begin{pmatrix}
0&0&0&0&0&2&1&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&2&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&1&0&0&2&0&0&0&0&0\\
\hline
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
\hline
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&1&1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&1&0&1&0&1&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&1&0&1&1&0&0\\
0&0&0&0&1&0&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1
\end{pmatrix}$$

The mod-2 support of this matrix is indeed EPT; this can be checked by reference to the tree given in Fig. 3. The edge associated with the ith degree equation is labeled d_i. The edge associated with the upper bound $x_{ij} \leq 1$ is labeled u_{ij}. Finally, next to the remaining three edges, we show the root and body of the corresponding light tooth inequality.

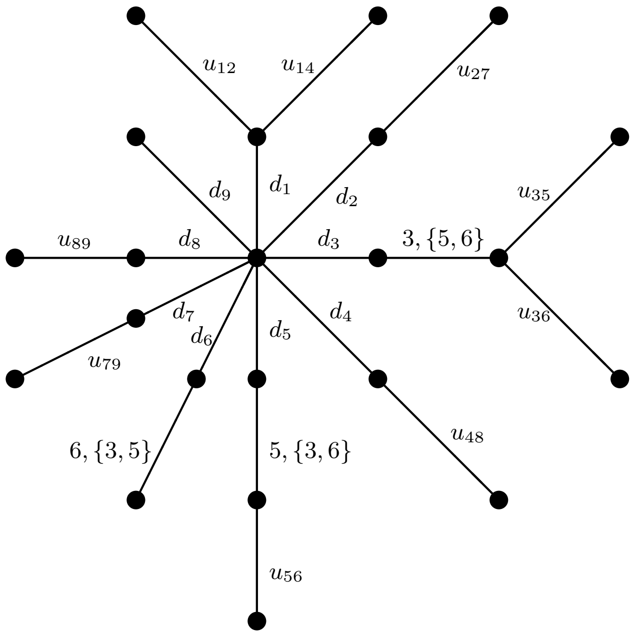

Fig. 3. Tree in demonstration of Theorem 4

Corollary 3. *If $x^* \in SEP(n)$, then a violated simple comb inequality which uses only light teeth can be found in polynomial time, if any exists.*

Proof: In the previous section we showed that the desired tooth inequalities are polynomial in number and that they can be found in polynomial time. The necessary system $Ax \leq b$ and its associated EPT matrix can easily be constructed in polynomial time. The result then follows from Theorem 2.

Example (Cont.): Applying stage 1 to the fractional point shown in Fig. 2, we find a violated simple comb inequality with $H = \{1, 2, 3\}$, $T_1 := \{1, 4\}$, $T_2 = \{2, 7\}$ and $T_3 = \{3, 5, 6\}$. The inequality is

$$x_{12} + x_{13} + x_{23} + x_{14} + x_{27} + x_{35} + x_{36} + x_{56} \leq 5,$$

and it is violated by one-third.

Now we need to deal with the second stage, i.e., the case where one of the teeth involved is heavy. From Lemma 3 we know that there are $\mathcal{O}(n^3)$ candidates for this heavy tooth. We do the following for each of these candidates: we take the collection of $\mathcal{O}(n^2)$ light tooth inequalities, and eliminate the ones whose teeth have a non-empty intersection with the heavy tooth under consideration. As we will show in the full version of this paper, the resulting modified matrix is still an EPT matrix, and the Caprara-Fischetti procedure can be repeated. This proves Theorem 1.

Now let us analyze the running time of this separation algorithm. First we consider stage 1. A careful reading of Caprara & Fischetti [2] shows that to separate $\{0, \frac{1}{2}\}$-cuts derived from a system $Ax \leq b$, where A is a $p \times q$ matrix, it is necessary to compute a minimum weight odd cut in a labeled weighted graph with $p - 1$ vertices and $p + q'$ edges, where q' is the number of variables which are currently non-zero. In our application, p is $\mathcal{O}(n^2)$, because there are $\mathcal{O}(n^2)$ light tooth inequalities; and q' is $\mathcal{O}(|E^*|)$. Therefore the graph concerned has $\mathcal{O}(n^2)$ vertices and $\mathcal{O}(n^2)$ edges.

Using the Padberg-Rao algorithm [29] to compute the minimum weight odd cut, and the pre-flow push algorithm [12] to solve the max-flow problems, stage 1 takes $\mathcal{O}(n^6 \log n)$ time. This is bad enough, but an even bigger running time is needed for stage 2, which involves essentially repeating the procedure used in stage 1 $\mathcal{O}(n^3)$ times. This leads to a running time of $\mathcal{O}(n^9 \log n)$ for stage 2, which, though polynomial, is totally impractical.

In the next section, we explore the potential for reducing this running time.

5 Improving the Running Time

To improve the running time, it suffices to reduce the number of 'candidates' for the teeth in a violated simple comb inequality. To this end, we now describe a simple lemma which enables us to eliminate teeth from consideration. We will then prove that, after applying the lemma, only $\mathcal{O}(|E^*|)$ *light* teeth remain.

Lemma 7. *Suppose a violated $\{0, \frac{1}{2}\}$-cut can be derived using the tooth inequality with root i and body S. If there exists a set $S' \subset V \setminus \{i\}$ such that*

- $E(i : S) \cap E^* = E(i : S') \cap E^*$,
- $|S'| - 2x^*(E(S')) - x^*(E(i : S')) \leq |S| - 2x^*(E(S)) - x^*(E(i : S))$,

then we can obtain a $\{0, \frac{1}{2}\}$-cut violated by at least as much by replacing the body S with the body S' (and adjusting the set of used non-negativity inequalities accordingly).

Proof: By Proposition 1, we have to consider the net change in the sum of the slacks of the *used* inequalities. The second condition in the lemma simply says that the slack of the tooth inequality with root i and body S' is not greater than the slack of the tooth inequality with root i and body S. Therefore replacing S with S' causes the sum of the slacks to either remain the same or decrease. Now we consider the *used* non-negativity inequalities. The only variables to receive an *odd* coefficient in a tooth inequality with root i and body S are those which correspond to edges in $E(i : S)$, and a similar statement holds for S'. So, for the edges in $E(i : (S \setminus S') \cup (S' \setminus S))$, the non-used non-negativity inequalities must now be used and vice-versa. But this has no effect on the sum of the slacks, because $E(i : (S \setminus S') \cup (S' \setminus S)) \subset E \setminus E^*$ by assumption and the slack of a non-negativity inequality for an edge in $E \setminus E^*$ is zero. Hence, the total sum of slacks is either unchanged or decreased and the new $\{0, \frac{1}{2}\}$-cut is violated by at least as much as the original.

A naive implementation of this idea runs in $\mathcal{O}(n^3|E^*|)$ time. The next theorem shows that, after it is applied, only $\mathcal{O}(|E^*|)$ *light* tooth inequalities remain.

Theorem 5. *After applying the elimination criterion of Lemma 7, at most* $4|E^*| - 2n$ *light tooth inequalities remain.*

Proof: Recall that, for any root i, the i-light sets form a nested family. Let S, S' be two distinct i-light sets remaining after the elimination criterion has been applied. If $S' \subset S$, then $E(i : (S \setminus S')) \cap E^* \neq \emptyset$. It is then easy to check that the total number of i-light sets is at most $2(d_i^* - 1)$, where d_i^* denotes the degree of i in G^*. So the total number of light tooth inequalities is at most $2\sum_{i \in V}(d_i^* - 1) = 4|E^*| - 2n$.

The effect on the running time of the overall separation algorithm is as follows. The graphs on which the minimum weight odd cuts must be computed now have only $\mathcal{O}(|E^*|)$ vertices and edges. Each odd cut computation now only involves the solution of $\mathcal{O}(|E^*|)$ max-flow problems. Thus, again using the pre-flow push max-flow algorithm, the overall running time of the separation algorithm is reduced from $\mathcal{O}(n^9 \log n)$ to $\mathcal{O}(n^3|E^*|^3 \log n)$. In practice, G^* is very sparse, so that this is effectively $\mathcal{O}(n^6 \log n)$.

It is natural to ask whether the running time can be reduced further. In our opinion further reduction is possible. First, we believe that a more complicated argument can reduce the number of *light* teeth to only $\mathcal{O}(n)$, and the number of *heavy* teeth to $\mathcal{O}(n|E^*|)$. Moreover, empirically we have found that, for each root i, it is possible to partition the set of i-heavy sets into $\mathcal{O}(d_i^*)$ nested families, each containing $\mathcal{O}(d_i^*)$ members (where d_i^* is as defined in the proof of Theorem 5). If this result could be proven to hold in general, we would be need to perform only $\mathcal{O}(|E^*|)$ minimum odd cut computations in stage 2. These changes would reduce the overall running time to only $\mathcal{O}(n^2|E^*|^2 \log(n^2/|E^*|))$, which leads us to hope that a practically useful implementation is possible. Further progress on this issue (if any) will be reported in the full version of the paper.

6 Concluding Remarks

We have given a polynomial-time separation algorithm for the simple comb inequalities, thus extending the result of Padberg and Rao [29]. This is the latest in a series of positive results concerned with comb separation (Padberg & Rao [29], Carr [5], Fleischer & Tardos [9], Caprara, Fischetti & Letchford [3], Letchford [21], Caprara & Letchford [4]).

A number of open questions immediately spring to mind. The main one is, of course, whether there exists a polynomial-time separation algorithm for *general* comb inequalities, or perhaps a generalization of them such as the *domino-parity* inequalities [21]. We believe that any progress on this issue is likely to come once again from the characterization of comb inequalities as $\{0, \frac{1}{2}\}$-cuts. Moreover, since separation of $\{0, \frac{1}{2}\}$-cuts is equivalent to the problem of finding a minimum weight member of a binary clutter [2], it might be necessary to draw on some of the deep decomposition techniques which are used in the theory of binary clutters and binary matroids (compare [2] with Grötschel & Truemper [17]).

Another interesting question is concerned with *lower bounds*. Let us call the lower bound obtained by optimizing over SEP(n) the *subtour bound*. The subtour bound is typically good in practice, and a widely held conjecture is that, when the edge costs satisfy the triangle inequality, the subtour bound is at least 3/4 of the optimal value. Moreover, examples are known which approach this value arbitrarily closely, see Goemans [11]. Now, using the results in this paper, together with the well-known equivalence of separation and optimization [14], we can obtain a stronger lower bound by also using the simple comb inequalities. Let us call this the *simple comb bound*. A natural question is whether the worst-case ratio between the simple comb bound and the optimum is greater than 3/4. Unfortunately, we know of examples where the ratio is still arbitrarily close to 3/4. Details will be given in the full version of the paper.

Finally, we can also consider special classes of graphs. For a given graph G, let us denote by SC(G) the polytope defined by the degree equations, the SECs, and the non-negativity and simple comb inequalities. (Now we only define variables for the edges in G.) Let us say that a graph G is *SC-perfect* if SC(G) is an integral polytope. Clearly, the TSP is polynomially-solvable on SC-perfect graphs. It would be desirable to know which graphs are SC-perfect, and in particular whether the class of SC-perfect graphs is closed under the taking of minors (see Fonlupt & Naddef [10]). Similarly, let us say that a graph is *SC-Hamiltonian* if SC(G) is non-empty. Obviously, every Hamiltonian graph is SC-Hamiltonian, but the reverse does not hold. (The famous *Peterson graph* is SC-Hamiltonian, but not Hamiltonian.) It would be desirable to establish structural properties for the SC-Hamiltonian graphs, just as Chvátal [6] did for the so-called *weakly Hamiltonian* graphs.

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