Integer Quadratic Quasi-polyhedra

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Abstract. This paper introduces two fundamental families of 'quasi-polyhedra' — polyhedra with a countably infinite number of facets — that arise in the context of integer quadratic programming. It is shown that any integer quadratic program can be reduced to the minimisation of a linear function over a quasi-polyhedron in the first family. Some fundamental properties of the quasi-polyhedra are derived, along with connections to some other well-studied convex sets. Several classes of facet-inducing inequalities are also derived. Finally, extensions to the mixed-integer case are briefly examined.

Keywords: mixed-integer non-linear programming, polyhedral combinatorics, convex analysis.

1 Introduction

In recent years, there has been increasing interest in *Mixed-Integer Non-Linear Programming* (MINLP), due to the realisation that it has a wealth of applications. This paper is concerned with a special case of MINLP: *Integer Quadratic Programming* (IQP). It is assumed that instances of IQP are written in the following standard form:

$$\min \left\{ c^T x + x^T Q x : \; Ax = b, \; x \in \mathbb{Z}^n_+ \right\} \quad , \tag{1}$$

where $c \in \mathbb{Q}^n$, $Q \in \mathbb{Q}^{n \times n}$, $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$. (As in linear programming, inequalities can be converted into equations using slack variables, and free variables can be expressed as the difference between two non-negative variables.)

We assume (without loss of generality) that the matrix Q is symmetric, but we do not require it to be positive semidefinite. That is, we do not assume that the objective function is convex.

Polyhedral combinatorics — the study of polyhedra associated with combinatorial problems — has proven to be a very useful tool for deriving strong formulations of *Mixed-Integer Linear Programs* (e.g., [1,16]). The purpose of this paper is to apply it to IQP. It turns out, however, that one has to deal with 'quasi-polyhedra': convex sets that are the intersection of a countably infinite number of half-spaces. For this reason, polyhedral theory has to be combined with elements of convex analysis. (A similar strategy was used in [5] to study a continuous quadratic optimisation problem.)

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The paper is structured as follows. In Sect. 2, two families of quasi-polyhedra are defined, and it is shown that any IQP instance can be reduced to the problem of optimising a linear function over a quasi-polyhedron in the first family. In Sect. 3, some fundamental properties of the quasi-polyhedra are derived, such as dimension, extreme points, affine symmetries, and relationships with some other well-studied convex sets. In Sect. 4, we derive several classes of valid inequalities, all of which are proven to induce facets under mild conditions. Finally, in Sect. 5, we suggest possible extensions to the mixed-integer case, and pose some questions for future research.

2 The Quasi-polyhedra

A standard trick when dealing with quadratic optimisation problems is to linearise the objective and/or constraints by introducing additional variables (e.g., [14,17,18]). More precisely, for $1 \le i \le j \le n$, we define a new variable y_{ij}, which represents the product $x_i x_j$. The IQP (1) can then be reformulated as:

$$\min \left\{ c^T x + q^T y : \ Ax = b, \ x \in \mathbb{Z}^n_+, \ y_{ij} = x_i x_j \ (1 \le i \le j \le n) \right\} \ ,$$

where $q \in \mathbb{Q}^{\binom{n+1}{2}}$ is defined appropriately. Notice that the non-linearity (and non-convexity, if any) is now captured in the constraints $y_{ij} = x_i x_j$.

It is an interesting fact that the linear equations can be eliminated from the problem. Indeed, we can delete an arbitrary linear equation $a^T x = r$, provided that we add $M(a^T x - r)^2$ to the objective function, where M is a suitable large integer. For this reason, one can concentrate on the unconstrained case, in which the linear system $Ax = b$ is vacuous.

The set of feasible solutions to an unconstrained IQP, in the extended (x, y)-space, is:

$$F^+_n := \left\{ (x, y) \in \mathbb{Z}_+^{n+\binom{n+1}{2}}, \ y_{ij} = x_i x_j \ (1 \le i \le j \le n) \right\} \ .$$

We wish to apply a polyhedral approach to IQP, and are therefore interested in the convex hull of this set. Unfortunately, there are two minor technical issues to address.

The first technical issue is that the convex hull of F^+_n is not closed, as expressed in the following proposition:

Proposition 1. *The convex hull of F^+_n is not closed for any n.*

Proof. For any $t \in \mathbb{Z}_+$, let (x^t, y^t) be the member of F^+_n that arises when $x_1 = t$, $y_{11} = t^2$, and all other variables are equal to zero. Moreover, let

$$(\tilde{x}^t, \tilde{x}^t) = \frac{1}{t^2}(x^t, y^t) + \frac{t^2 - 1}{t^2}(x^0, y^0) \ .$$

Note that $(\tilde{x}^t, \tilde{x}^t)$ is a convex combination of members of F^+_n and therefore lies in the convex hull. Note also that $(\tilde{x}^t, \tilde{x}^t)$ is obtained by setting $x_1 = 1/t$, $y_{11} = 1$, and all other variables to zero. On the other hand, the point with $y_{11} = 1$ and all other variables at zero does not lie in the convex hull of F_n. Since the convex hull does not contain all of its limit points, it is not closed. □

We are therefore led to look at the closure of the convex hull, which we denote by IQ_n^+. Figure 1 represents IQ_1^+. It can be seen that it is described by the non-negativity inequality $x_1 \geq 0$, together with the inequalities $y_{11} \geq (2t+1)x_1 - t(t+1)$ for all $t \in \mathbb{Z}_+$. (A similar observation was made by Michaels & Weismantel [11] for a closely-related family of polytopes.)

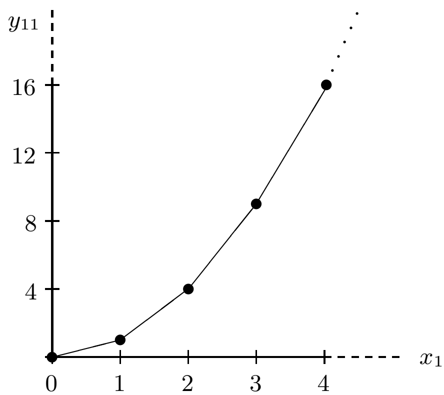

Fig. 1. The convex set IQ_1^+

The second technical issue is that IQ_n^+ is, in fact, not a polyhedron. A polyhedron is defined as the intersection of a *finite* number of half-spaces, but we have seen that IQ_1^+ is the intersection of a *countably infinite* number of half-spaces. (The results that we give in Sect. 4 show that the same holds when $n > 1$ as well.) The correct term for such sets is *quasi-polyhedra* (see, e.g., Anderson *et al.* [2]). Fortunately, this issue does not cause any difficulty in what follows.

For the purposes of what follows, we introduce a closely-related family of quasi-polyhedra, obtained by omitting the non-negativity requirement. Specifically, we define

$$F_n := \left\{ (x,y) \in \mathbb{Z}^{n+\binom{n+1}{2}},\ y_{ij} = x_i x_j \ (1 \leq i \leq j \leq n) \right\} \ ,$$

and then let IQ_n denote the closure of the convex hull of F_n. One can check that IQ_1 is described by the inequalities $y_{11} \geq (2t+1)x_1 - t(t+1)$ for all $t \in \mathbb{Z}$.

Next, we present two simple complexity results:

Proposition 2. *Minimising a linear function over* IQ_n^+ *is* $\mathcal{NP}$*-hard in the strong sense.*

Proof. It follows from the above discussion that this problem is equivalent to IQP. Now, IQP is clearly $\mathcal{NP}$-hard in the strong sense, since it contains Integer Linear Programming as a special case. □

Proposition 3. *Minimising a linear function over* IQ_n *is* $\mathcal{NP}$*-hard in the strong sense.*

Proof. The well-known *Closest Vector Problem*, proven to be strongly $\mathcal{NP}$-hard by van Emde Boas [8], takes the form:

$$\min\left\{\|Bx-t\|_2 : \, x \in \mathbb{Z}^n\right\},$$

where $B \in \mathbb{Z}^{n\times n}$ is a basis matrix and $t \in \mathbb{Q}^n$ is a target point. Clearly, squaring the objective function leaves the optimal solution unchanged. The resulting problem is one of minimising a quadratic function over the integer lattice $\mathbb{Z}^n$. It follows from the definitions that this is equivalent to minimising a linear function over IQ_n. □

We therefore cannot expect to obtain complete linear descriptions of IQ_n or IQ_n^+ for general n.

On a more positive note, we have the following result:

Proposition 4. *Minimising a linear function over* IQ_n *is solvable in polynomial time when* n *is fixed.*

Proof. As already mentioned, minimising a linear function over IQ_n is equivalent to minimising a quadratic function over the integer lattice $\mathbb{Z}^n$. Now, if the function is not convex, the problem is easily shown to be unbounded. If, on the other hand, the function is convex, then the problem can be solved for fixed n by the algorithm of Khachiyan & Porkolab [9]. □

There is therefore some hope of obtaining a complete linear description of IQ_n for small values of n (just as we have already done for the case $n = 1$). We do not know the complexity of minimising a linear function over IQ_n^+ for fixed n.

3 Fundamental Properties of the Quasi-polyhedra

In this section, we establish some fundamental properties of the quasi-polyhedra IQ_n and IQ_n^+.

3.1 Dimension and Extreme Points

We begin with two elementary results:

Proposition 5. *For all* n*, both* IQ_n *and* IQ_n^+ *are full-dimensional, i.e., of dimension* $n + \binom{n+1}{2}$*.*

Proof. Consider the following extreme points of IQ_n^+:

- the origin (i.e., all variables set to zero);
- for $i = 1, \ldots, n$, the point having $x_i = y_{ii} = 1$ and all other variables zero;
- for $i = 1, \ldots, n$, the point having $x_i = 2$, $y_{ii} = 4$ and all other variables zero;
- for $1 \le i < j \le n$, the point having $x_i = x_j = 1$, $y_{ii} = y_{jj} = y_{ij} = 1$, and all other variables zero.

These $n + \binom{n+1}{2} + 1$ points are easily shown to be affinely independent, and therefore IQ_n^+ is full-dimensional. Since IQ_n^+ is contained in IQ_n, the same is true for IQ_n. □

Proposition 6. *Every point in F_n is an extreme point of IQ_n, and every point in F_n^+ is an extreme point of IQ_n^+.*

Proof. Let $\bar{x}$ be an arbitrary point in $\mathbb{Z}^n$, and let $(\bar{x}, \bar{y})$ be the corresponding member of F_n. The quadratic function $\sum_{i=1}^n (x_i - \bar{x}_i)^2$ has a unique minimum at $x = \bar{x}$. Since every point in F_n satisfies $y_{ij} = x_i x_j$ for all $1 \leq i \leq j \leq n$, the *linear* function $\sum_{i=1}^n (y_{ii} - 2\bar{x}_i x_i + \bar{x}_i^2)$ has a unique minimum at $(\bar{x}, \bar{y})$. Therefore $(\bar{x}, \bar{y})$ is an extreme point of IQ_n. The proof for IQ_n^+ is similar. □

3.2 Affine Symmetries

Now we examine the affine symmetries of the quasi-polyhedra, i.e., affine transformations that map the quasi-polyhedra onto themselves.

Proposition 7. *Let π be an arbitrary permutation of the index set $\{1, \ldots, n\}$. Consider the linear transformation that takes any $(x, y) \in \mathbb{R}^{n+\binom{n}{2}}$ and maps it to a point $(x', y') \in \mathbb{R}^{n+\binom{n}{2}}$, where*

- *$x'_i = x_{\pi(i)}$ for all $i \in \{1, \ldots, n\}$,*
- *$y'_{ij} = y_{\pi(i),\pi(j)}$ for all $1 \leq i \leq j \leq n$.*

This transformation maps IQ_n^+ onto itself.

Proof. Trivial. □

Theorem 1. *Let U be a unimodular integral square matrix of order n, and let $w \in \mathbb{Z}^n$ be an arbitrary integer vector. Consider the affine transformation that takes any $(x, y) \in \mathbb{R}^{n+\binom{n}{2}}$ and maps it to a point $(x', y') \in \mathbb{R}^{n+\binom{n}{2}}$, where*

- *$x' = Ux + w$;*
- *$y'_{ij} = x'_i x'_j$ for all $1 \leq i \leq j \leq n$.*

This transformation maps IQ_n onto itself.

Proof. Let (x, y) be an extreme point of IQ_n, and let (x', y') be its image under the transformation. Since U and w are integral, x' is integral. Moreover, since $y'_{ij} = x'_i x'_j$, (x', y') is an extreme point of IQ_n. For the reverse direction, let (x', y') be an extreme point of IQ_n, and let (x, y) be its image under the inverse transformation. Note that $x = U^{-1}(x' - w)$, and is therefore integral. Moreover, $y_{ij} = x_i x_j$ for all $1 \leq i \leq j \leq n$, which implies that (x, y) is an extreme point of IQ_n. □

Proposition 7 simply states that IQ_n^+ is invariant under a permutation of the index set $\{1, \ldots, n\}$, which is unsurprising. Theorem 1, on the other hand, has a very useful corollary:

Corollary 1. *Let U be a unimodular integral square matrix of order n, and let $w \in \mathbb{Z}^n$ be an arbitrary integer vector. If the linear inequality $\alpha^T x + \beta^T y \geq \gamma$ is facet-inducing for* IQ_n*, then so is the linear inequality $\alpha^T x' + \beta^T y' \geq \gamma$, where (x', y') is defined as in Theorem 1.*

Intuitively speaking, this means that any inequality inducing a facet of IQ_n can be 'rotated' and 'translated' to yield a countably infinite family of facet-inducing inequalities.

It is also possible to convert any facet-inducing inequality for IQ_n into a facet-inducing inequality for IQ_n^+:

Theorem 2. *Suppose the inequality $\alpha^T x + \beta^T y \geq \gamma$ induces a facet of* IQ_n*. Then there exists a vector $v \in \mathbb{Z}_+^n$ such that the inequality $\alpha^T x' + \beta^T y' \geq \gamma$ induces a facet of* IQ_n^+*, where:*

- *$x' = x - v$;*
- *$y'_{ij} = x'_i x'_j$ for all $1 \leq i \leq j \leq n$.*

Proof. Let $d = n + \binom{n+1}{2}$. Since the original inequality $\alpha^T x + \beta^T y \geq \gamma$ induces a facet of IQ_n, there exist d affinely-independent members of F_n that satisfy it at equality. Let $x^1, \ldots, x^d$ denote the corresponding x vectors. Now, for $i = 1, \ldots, n$, set v_i to $\min_{1 \leq j \leq d} x_i^j$. The resulting transformed inequality $\alpha^T x' + \beta^T y' \geq \gamma$ induces a facet of IQ_n by Theorem 1. It is also valid for IQ_n^+, since IQ_n^+ is contained in IQ_n. Moreover, the points $x^1 - v, \ldots, x^d - v$ all lie in the non-negative orthant by construction. These points correspond to affinely-independent members of F_n^+ that satisfy the transformed inequality at equality. Therefore the transformed inequality induces a facet of IQ_n^+. □

Therefore, any inequality inducing a facet of IQ_n yields a countably infinite family of facet-inducing inequalities for IQ_n^+ as well.

3.3 Two Related Cones

Recall that a symmetric matrix $M \in \mathbb{R}^{n \times n}$ is called *positive semidefinite* (psd) if it can be factorised as AA^T for some real matrix A. The set of psd matrices of order n forms a convex cone in $\mathbb{R}^{n \times n}$. It is well known that this cone is completely described by the linear inequalities $v^T M v \geq 0$ for all vectors $v \in \mathbb{R}^n$.

We now use a standard construction [10,17] to establish a connection between IQ_n and the psd cone. Define the $n \times n$ symmetric matrix $Y = xx^T$, and note that, for any $1 \leq i \leq j \leq n$, $Y_{ij} = y_{ij}$. Define also the augmented matrix

$$\hat{Y} := \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T = \begin{pmatrix} 1 & x^T \\ x & Y \end{pmatrix} .$$

Since $\hat{Y}$ is the product of a vector and its transpose, it must be psd. Equivalently, $v^T Y v + (2s) v^T x + s^2 \geq 0$ for all vectors $v \in \mathbb{R}^n$ and scalars $s \in \mathbb{R}$. This observation immediately yields the following result:

Proposition 8. *The following 'psd inequalities' are valid for* IQ_n *(and therefore also for* IQ_n^+*):*

$$(2s)v^T x + \sum_{i=1}^{n} v_i^2 y_{ii} + 2 \sum_{1 \le i < j \le n} v_i v_j y_{ij} + s^2 \ge 0 \qquad (\forall v \in \mathbb{R}^n, s \in \mathbb{R}) \ . \quad (2)$$

To the knowledge of the author, the validity of the psd inequalities for extended formulations of quadratic optimisation problems was first observed by Ramana [15]. The inequalities can be shown to induce proper faces of IQ_n and IQ_n^+ under mild conditions. We will see in the next section, however, that they never induce facets.

Now recall that a symmetric matrix $M \in \mathbb{R}^{n \times n}$ is called *completely positive* if it can be factorised as AA^T for some *non-negative* real matrix A. The set of completely positive matrices of order n also forms a convex cone in $\mathbb{R}^{n \times n}$. Using exactly the same argument as above, any valid inequality for the completely positive cone yields a valid inequality for IQ_n^+. Unfortunately, this additional information does not help us much, because a complete linear description of the completely positive cone is unknown, and unlikely to be found for general n [12].

3.4 A Connection to the Boolean Quadric Polytope

We close this section by pointing out a connection between IQ_n, IQ_n^+ and the so-called *boolean quadric polytope*. The boolean quadric polytope of order n is denoted by BQP_n and is defined as:

$$\mathrm{BQP}_n = \mathrm{conv}\left\{(x,y) \in \{0,1\}^{n+\binom{n}{2}} : y_{ij} = x_i x_j \ (1 \le i < j \le n)\right\} \ .$$

Note that the y_{ii} variables are not present in the case of BQP_n.

The boolean quadric polytope was defined by Padberg [14] in the context of quadratic 0-1 programming. It has many applications in other fields and has been studied in great depth [7].

We will need the following lemma:

Lemma 1. *For all* $1 \le i \le n$*, the inequality* $y_{ii} \ge x_i$ *is valid for* IQ_n*.*

Proof. This follows from the fact that all members of F_n satisfy $y_{ii} = x_i^2$, and the fact that $t^2 \ge t$ for any integer t. □

The following proposition states that BQP_n is essentially nothing but a face of IQ_n:

Proposition 9. *Let* H *be the face of* IQ_n *obtained by setting the inequality* $y_{ii} \ge x_i$ *to an equation for all* $1 \le i \le n$*. The boolean quadric polytope* BQP_n *is an affine image of* H*.*

Proof. Note that $t^2 = t$ if and only if $t \in \{0,1\}$. Therefore, the extreme points of H are precisely the members of F_n that satisfy $x \in \{0,1\}^n$. So there is a one-to-one correspondence between extreme points of H and extreme points of BQP_n. Moreover, every extreme point (x^*, y^*) of BQP_n can be mapped onto an extreme point of H simply by setting $y^*_{ii} = x^*_i$ for all $i = 1, \ldots, n$. This mapping is affine. □

An immediate consequence of Proposition 9 is that valid or facet-inducing inequalities for BQP_n can be *lifted* to yield valid or facet-inducing inequalities for IQ_n:

Corollary 2. *Suppose the inequality*

$$\sum_{i=1}^{n} a_i x_i + \sum_{1 \le i < j \le n} b_{ij} y_{ij} \le c$$

induces a facet of BQP_n*. Then there exists at least one facet-inducing inequality for* IQ_n *of the form*

$$\sum_{i=1}^{n} (a_i - \lambda_i) x_i + \sum_{i=1}^{n} \lambda_i y_{ii} + \sum_{1 \le i < j \le n} b_{ij} y_{ij} \le c \ ,$$

with $\lambda \in \mathbb{Q}^n$*.*

Similar results can be shown to hold for IQ_n^+.

4 Some Facet-Inducing Inequalities

We now move on to consider some specific classes of facet-inducing inequalities.

4.1 Non-negativity Inequalities

Since IQ_n^+ is contained in the completely positive cone, it is clear that all variables are constrained to be non-negative. The following theorem states conditions under which non-negativity inequalities induce facets of IQ_n^+:

Theorem 3. *The inequalities* $x_i \ge 0$ *for all* $1 \le i \le n$*, and the inequalities* $y_{ij} \ge 0$ *for all* $1 \le i < j \le n$*, induce facets of* IQ_n^+*. The inequalities of the form* $y_{ii} \ge 0$*, on the other hand, never induce facets of* IQ_n^+*.*

Proof. To see that the inequalities of the form $y_{ij} \ge 0$ induce facets, simply note that all but one of the affinely-independent points listed in the proof of Proposition 5 satisfy $y_{ij} = 0$. To see that the inequalities of the form $y_{ii} \ge 0$ do not induce facets, simply note that they are dominated by the inequalities $x_i \ge 0$ and $y_{ii} \ge x_i$ (refer to Fig. 1). The inequalities of the form $x_i \ge 0$ are a little more tricky: one can easily construct $n + \binom{n}{2}$ affinely-independent points with $x_i = 0$, but to complete the proof one needs an additional n extreme rays of IQ_n^+ having $x_i = 0$. The proof of Proposition 1 shows that there is a ray with $y_{ii} = 1$ and all other variables zero. Using a similar argument, one can show that, for all $j \ne i$, there is a ray with $x_j = y_{ij} = 1$ and all other variables zero. □

The non-negativity inequalities are of course not valid for IQ_n.

4.2 Split Inequalities

In this subsection, we introduce a more interesting class of inequalities, valid for both IQ_n^+ and IQ_n. Before presenting them, we recall the definition of *split disjunctions*, taken from [6]. A split disjunction is a disjunction of the form $(v^T x \le s) \vee (v^T \ge s+1)$, where $v \in \mathbb{Z}^n$ and $s \in \mathbb{Z}$. Split disjunctions are obviously satisfied by all lattice points $x \in \mathbb{Z}^n$. An example of a split disjunction is illustrated in Fig. 2.

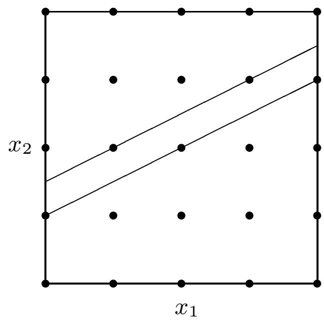

Fig. 2. The split disjunction $(x_1 - 2x_2 \le -2) \vee (x_1 - 2x_2 \ge -1)$

The following proposition uses split disjunctions to derive an infinite family of valid inequalities:

Proposition 10. *For any vector $v \in \mathbb{Z}^n$ and scalar $s \in \mathbb{Z}$, the following 'split' inequality is valid for both IQ_n and IQ_n^+:*

$$(2s+1)v^T x + \sum_{i=1}^{n} v_i^2 y_{ii} + 2 \sum_{1 \le i < j \le n} v_i v_j y_{ij} + s(s+1) \ge 0 \ . \tag{3}$$

Proof. The split disjunction $(v^T x \le -s-1) \vee (v^T x \ge -s)$ implies the quadratic inequality $(v^T x + s)(v^T x + s + 1) \ge 0$. Expanding this and substituting Y for xx^T yields $v^T Y v + (2s+1)v^T x + s(s+1) \ge 0$, which is equivalent to the inequality (3). □

We remark that an important class of cutting planes for Mixed-Integer Linear Programs, called *split cuts*, can be derived using split disjunctions [3,6]. It is important to note however that the split inequalities (3) are *not* split cuts in the traditional sense. Indeed, split cuts arise from the interaction between a split disjunction and a set of linear constraints, whereas the split inequalities (3) are directly implied by the disjunctions themselves.

One can check that IQ_1 is completely described by the split inequalities, and that IQ_1^+ is completely described by the split inequalities together with the non-negativity inequality $x_1 \ge 0$. The following three theorems give further evidence that split inequalities are theoretically strong:

Theorem 4. *The split inequalities (3) dominate the psd inequalities (2).*

Proof. First, suppose that a psd inequality is derived using an integral vector v and an integral scalar s. Recall that the psd inequality can be written as $v^TYv + (2s)v^Tx + s^2 \geq 0$. This is dominated by the two inequalities $v^TYv + (2s+1)v^Tx + s(s+1) \geq 0$ and $v^TYv + (2s-1)v^Tx + s(s-1) \geq 0$, which are both split inequalities.

To complete the proof, we must show that the psd inequalities derived from integral v and s dominate all the others. Suppose a point (x^*, y^*) violates a psd inequality with non-integral v or s, and let ϵ be a small positive quantity. Let v' be a rational vector such that $|v'_i - v_i| < \epsilon$ for all i, and let s' be a rational number such that $|s' - s| < \epsilon$. Provided ϵ is small enough, the psd inequality obtained by using v' and s' in place of v and s will also be violated by (x^*, y^*). Now let M be a positive integer such that $Mv' \in \mathbb{Z}^n$ and $Ms' \in \mathbb{Z}$. The psd inequality with Mv' and Ms' in place of v' and s' will also be violated by (x^*, y^*). Therefore the original psd inequality is redundant. □

Theorem 5. *Split inequalities induce facets of* IQ_n *if the non-zero components of* v *are relatively prime.*

Proof. First, note that the trivial inequality $y_{11} \geq x_1$ is a split inequality, obtained by linearising the quadratic inequality $(x_1 - 1)x_1 \geq 0$. This trivial split inequality induces a facet of IQ_n, because all but one of the affinely-independent points listed in the proof of Proposition 5 satisfy $y_{11} = x_1$.

Now consider a non-trivial split inequality of the form (3), and assume that the non-zero components of v are relatively prime. A well-known result on integral matrices (see, e.g., p. 15 of Newman [13]) implies that there exists a unimodular matrix $U \in \mathbb{Z}^{n \times n}$ having v as its first row. Let U be such a matrix, and let $w \in \mathbb{Z}^n$ be an arbitrary vector satisfying $w_1 = s + 1$. Note that, if (x, y) is an extreme point of IQ_n and (x', y') is the transformed extreme point described in Theorem 1, then $x'_1 = v^Tx + s + 1$ and $y'_{11} = (x'_1)^2 = v^TYv + 2(s+1)v^Tx + (s+1)^2$. Thus, if we apply the transformation mentioned in Corollary 1 to the trivial split inequality $y_{11} \geq x_1$, we obtain the inequality $v^TYv + 2(s+1)v^Tx + (s+1)^2 \geq v^Tx + s + 1$. This is equivalent to the non-trivial split inequality. By Corollary 1, it induces a facet of IQ_n. □

Theorem 6. *Split inequalities induce facets of* IQ_n^+ *if the non-zero components of* v *are relatively prime and not all of the same sign.*

Proof. First, note that when v satisfies the stated condition, there exists a vector $w \in \mathbb{Z}^n$ such that $v^Tw = 0$ and such that $w_i > 0$ for all i. To see this, let k and k' be the number of components of v that are positive and negative, respectively, and let m be the product of the non-zero components of v. The desired vector w can be obtained by setting w_i to $k'|m|/v_i$ when $v_i > 0$, to $k|m|/|v_i|$ when $v_i < 0$, and to 1 otherwise.

Second, observe that an extreme point $(\bar{x}, \bar{y})$ of IQ_n satisfies the split inequality (3) at equality if and only if $v^T\bar{x} \in \{-s-1, -s\}$. Therefore, if $(\bar{x}, \bar{y})$ is such an extreme point, then so is the extreme point obtained by replacing $\bar{x}$ with $\bar{x}+w$, and adjusting $\bar{y}$ accordingly. Let us call this (affine) transformation 'shifting'.

Now, since the split inequality induces a facet of IQ_n under the stated conditions, there exist $n+\binom{n+1}{2}$ affinely-independent points in F_n that satisfy the split inequality at equality. By shifting this set of points, repeatedly if necessary, we obtain $n+\binom{n+1}{2}$ affinely-independent points in F_n^+ that satisfy the split inequality at equality. Therefore the split inequality induces a facet of IQ_n^+ as well. □

If the non-zero components of the vector v all have the same sign, then the split inequality need not induce even a proper face of IQ_n^+ (because there may not exist a lattice point $x \in \mathbb{Z}_+^n$ such that $v^T x \in \{-s-1, -s\}$). Theorem 2 implies however the following result:

Corollary 3. *Let $v \in \mathbb{Z}^n$ be such that all its components are relatively prime and of the same sign. Then there exists an integer s, of the opposite sign, such that the split inequality (3) induces a facet of IQ_n^+.*

To close this subsection, we remark that Propositions 9 and 10 imply the validity of the following inequalities for BQP_n:

$$\sum_{i=1}^{n} v_i(v_i+2s+1)x_i + 2\sum_{1\le i<j\le n} v_i v_j y_{ij} + s(s+1) \ge 0 \qquad (\forall v \in \mathbb{Z}^n, s \in \mathbb{Z}) \ . \quad (4)$$

These inequalities were discovered by Boros & Hammer [4].

4.3 Other Inequalities

We have seen that IQ_1^+ is completely described by the split inequalities and the non-negativity inequality $x_1 \ge 0$. A natural question is whether the split and non-negativity inequalities are enough to describe IQ_2^+. This is unfortunately not the case, as we now explain.

Consider the two lines in $\mathbb{R}^2$ defined by the equations $x_1+x_2=3$ and $x_1+2x_2=4$. As illustrated in Fig. 3, these lines pass through several points in $\mathbb{Z}_+^2$. Moreover, all points in $\mathbb{Z}_+^2$ are either above both lines (satisfying $x_1+x_2 \ge 3$ and $x_1+2x_2 \ge 4$), or below both lines (satisfying $x_1+x_2 \le 3$ and $x_1+2x_2 \le 4$). This implies that all points in F_2^+ satisfy the non-linear inequality $(x_1+x_2-3)(x_1+2x_2-4) \ge 0$. This implies that the linear inequality

$$-7x_1 - 9x_2 + y_{11} + 3y_{12} + 2y_{22} \ge 12$$

is valid for IQ_2^+. One can check (either by hand or with the aid of a computer) that this inequality induces a facet of IQ_2^+.

Using 'non-standard' split disjunctions of this kind, one can easily derive other inequalities that induce facets of IQ_n^+ for $n \ge 2$. Details will be given in the full version of the paper.

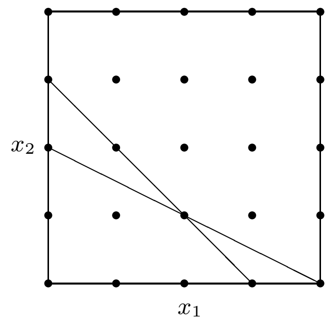

Fig. 3. A 'non-standard' split when $n = 2$

Turning attention to IQ_n, we have seen that IQ_1 is completely described by split inequalities. A natural question is whether the split inequalities are enough to describe IQ_2. We do not know the answer to this question. We are however able to show that the split inequalities do not completely describe IQ_6. Indeed, one can show using results in [7] that the boolean quadric polytope BQP_6 is not completely described by the Boros-Hammer inequalities (4). This implies, via Corollary 2, that there exist facet-inducing inequalities for IQ_6 that are not split inequalities. Specific inequalities of this kind will be presented in the full version of the paper.

5 Concluding Remarks

This paper marks a first step in applying polyhedral methods to Integer Quadratic Programs. There are many interesting open questions. We have already mentioned the question of whether one can optimise a linear function over IQ_n^+ in polynomial time for fixed n, and whether the split inequalities completely describe IQ_2. Another important question is whether the separation problem for the split inequalities can be solved in polynomial time.

Perhaps more importantly, it would be worthwhile extending the approach given in this paper to the mixed-integer case. Some preliminary observations on this case are the following. First, one has to deal with general convex sets rather than quasi-polyhedra, since the number of feasible solutions is no longer countable. Second, the split inequalities should be defined only when the components of the vector v are zero for all continuous variables, since otherwise they may not be valid. Third, it is no longer the case that the psd inequalities are dominated by the split inequalities. Indeed, if the vector v has a non-zero component for at least one continuous variable, it is even possible for a psd inequality to induce a maximal face of the convex set. Details will be given in the full version of the paper.

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