1 Programming Cumulative Normals on the HP12C

1.1 The numerical problem

$$\begin{aligned}
\mathrm{erf}\,(z) &= \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n\,z^{2n+1}}{n!\,(2n+1)} \\
&= \frac{2e^{-z^2}}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{2^n z^{2n+1}}{1\times 3\times ...\times(2n+1)} \\
\mathrm{erf}\,(-z) &= -\,\mathrm{erf}\,(z)\,;\;\; \mathrm{erf}\,(-\infty) = -1;\;\; \mathrm{erf}\,(0) = 0;\;\; \mathrm{erf}\,(\infty) = 1
\end{aligned}$$

$$\begin{aligned}
N\,(d) &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}s^2}ds = \frac{1}{2}+\frac{1}{2}\,\mathrm{erf}\left(z=\frac{d}{\sqrt{2}}\right) \\
N\,(-d) &= 1-N\,(d)\,;\;\; N\,(-\infty)=0;\;\; N\,(0)=0.5;\;\; N\,(\infty)=1
\end{aligned}$$

1.2 The numerical algorithm

See Table 1.

1.3 Notes

- Use a tolerance of about 0.001 (STO 0), lower values generate more iterations.
- Can produce values (slightly) less than zero or greater than one depending on tolerance!
- Other series may be easier to sum.
- This programme is not as short as possible, it has been written for ease of presentation (try and write a shorter or faster programme?).
- Having π in the programme consumes 11 programme lines but frees up a register (equivalent to 7 programme lines).
- Analytic (easier to calculate) approximations are also available.

1.4 Cumulative Normal Tables

See Table 2.

$$N\,(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{1}{2}s^2}ds$$

Lines	Codes	Keystokes	Function and action
00			blank start line
01	2	2	take d and conv. to erf z
02	43 21	g y^x ($\sqrt{x}$)	$\sqrt{2}$
03	10	$\div$	$z = d/\sqrt{2}$
04	44 1	STO 1	store z in 1
05	44 3	STO 3	also store z in cumulative sum 3
06	0	0	reset n (summation index) to zero
07	44 2	STO 2	
08	45 2	RCL 2	loop point
09–10	1 40	1 +	increment n by one and store
11	44 2	STO 2	
12	45 1	RCL 1	recall z
13	45 2	RCL 2	recall n
14–17	2 20 1 40	2 $\times$ 1 +	$2n + 1$
18	21	y^x	z^{2n+1}
19–20	1 16	1 CHS	-1
21	45 2	RCL 2	n
22	21	y^x	$(-1)^n$
23	20	$\times$	$z^{2n+1}(-1)^n$
24	45 2	RCL 2	n
25	43 3	g 3 ($n!$)	$n!$
26	10	$\div$	$z^{2n+1}(-1)^n/n!$
27	45 2	RCL 2	n
28–31	2 20 1 40	2 $\times$ 1 +	$2n + 1$
32	10	$\div$	$z^{2n+1}(-1)^n/(2n+1)/n!$
33	36	ENTER	push up into stack for later use (tol.)
34	36	ENTER	push up into stack for later use (sum)
35	45 3	RCL 3	recall sum to date
36	40	+	add partial sum (still one more in stack)
37	44 3	STO 3	store in 3
38	33	R↓	bring down partial sum from stack
39–40	2 21	2 y^x	square it and then ...
41	43 21	g y^x ($\sqrt{x}$)	...square root it (take the absolute value)
42	45 0	RCL 0	recall tolerance (0.001 is reasonable)
43	43 34	g $x \gtrless y$ ($x \leqslant y$)	compare abs. value of partial sum to tol.
44	43 33 08	g R↓ 08 (GTO 08)	if greater keep going, if not terminate
45	45 3	RCL 3	recall total sum from 3
46–56	3 48 1 4 1 5 9 2 6 5 4	3 . 1 4 1 5 9 2 6 5 4	π in11 steps (could be in memory for 7)
57	43 21	g y^x ($\sqrt{x}$)	$\sqrt{\pi}$
58	10	$\div$	$\frac{1}{\sqrt{\pi}} \sum_{n=0}^{(tol)} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}$
59–62	0 48 5 40	0.5 +	$N(d) = 0.5 + \frac{1}{\sqrt{\pi}} \sum_{n=0}^{(tol)} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}$

Table 1: HP12C programme to evaluate cumulative normal integrals $N(d)$ by summing the series $erf(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}$.

d	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91308	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
5.0	1.00000									

Table 2: Cumulative normal values $N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}s^2} ds$ for values of d from 0 to 4 in steps of 0.01. Use linear interpolation for interim and $N(-d) = 1 - N(d)$ for negative values.