Joint modelling of the bulk and tail of bivariate data

Lídia André

Jennifer Wadsworth and Adrian O'Hagan

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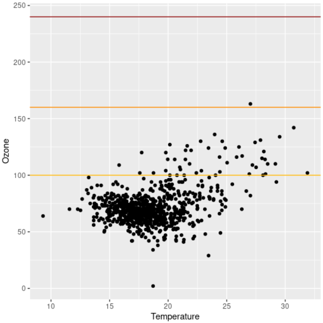
Motivation

Interest not only in the extremes but also the bulk of the distribution - *e.g.* environmental applications



Univariate Framework

There have been proposed parametric, semi-parametric and non-parametric models

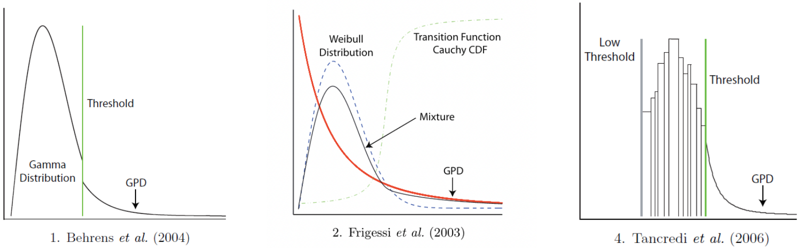


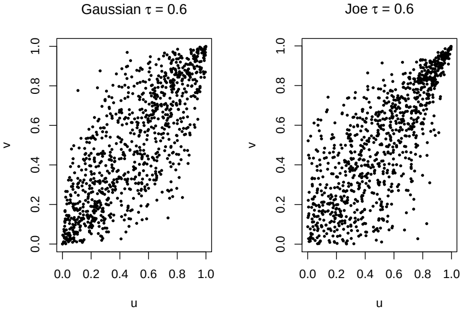
Figure 1: Taken from Scarrott and MacDonald (2012)

Copulas

In a multivariate setting we are also concerned about the dependence between variables.

A copula C is a joint distribution of a random vector $(X_1, \ldots, X_d)$

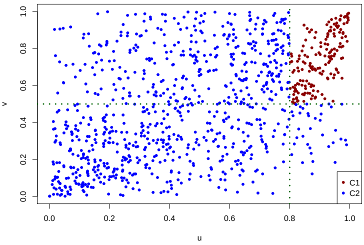
$$F(x_1, \ldots, x_d) = C\left(F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\right), \quad d \geq 2$$



Multivariate Framework

Aulbach et al. (2012) model the full data set by fitting one copula to the body and another to the upper tail

- It sometimes doesn't offer a smooth transition between the two copulas
- It requires the choice of thresholds
- The likelihood of the model doesn't have a closed form so no inference was done



Weighted Copula Model

For $(u^*, v^*) \in [0,1]^2$, we define the density c^* as

$$c^*(u^*, v^*; \gamma) = \frac{\pi(u^*, v^*; \theta)c_t(u^*, v^*; \boldsymbol{\alpha}) + [1 - \pi(u^*, v^*; \theta)]c_b(u^*, v^*; \boldsymbol{\beta})}{K(\gamma)}$$

[1] For more details see André et al. (2023)

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- $c_t, \ c_b \rightarrow$ copula densities tailored to the tail and body, respectively.

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- $c_t, \; c_b \rightarrow$ copula densities tailored to the tail and body, respectively.
- $\pi(u^*, v^*; \theta) \rightarrow$ dynamic weighting function, defined in $[0,1]^2$ and increasing in u^* and v^*

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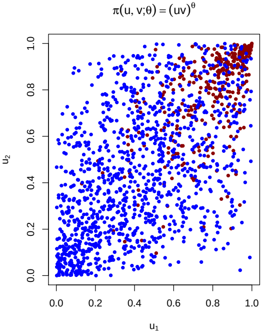
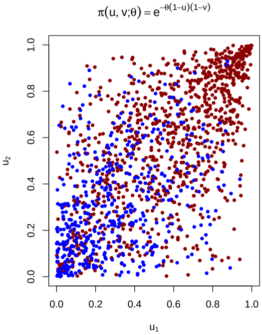
- $c_t,\ c_b \rightarrow$ copula densities tailored to the tail and body, respectively.
- $\pi(u^*, v^*; \theta) \rightarrow$ dynamic weighting function, defined in $[0,1]^2$ and increasing in u^* and v^*
- $\boldsymbol{\gamma} = (\theta, \boldsymbol{\alpha}, \boldsymbol{\beta}) \rightarrow$ vector of model parameters
- $K(\boldsymbol{\gamma}) \rightarrow$ normalising constant [1]

[1] For more details see André et al. (2023)

Weighted Copula Model

- Doesn't require a choice of threshold
- Offers a smooth transition between the body and tail copulas
- However, it is also hard to perform inference on it





Inference

The inference on the model was achieved by fitting the copula of the density c^* via numerical integration as follows

$$c(u, v; \gamma) = \frac{c^* \left(F_{U^*}^{-1}(u), F_{V^*}^{-1}(v); \gamma \right)}{f_{U^*} \left(F_{U^*}^{-1}(u) \right) f_{V^*} \left(F_{V^*}^{-1}(v) \right)}$$

where

$$F_{U^*}(u^*) = P[U^* \leq u^*] = \int_0^{u^*} \int_0^1 c^*(u, v) \mathrm{d}v \mathrm{d}u$$

$$f_{U^*}(u^*) = \int_0^1 c^*(u^*, v) \, \mathrm{d}v, \qquad v \in (0, 1)$$

Extremal Dependence Properties

It is important to know if extreme values of the variables are likely to occur together (**asymptotic dependence**) or not (**asymptotic independence**)

$$\chi = \lim_{r \to 1} P\left[F_Y(y) > r \mid F_X(x) > r\right],$$

$$P\left[F_Y(y) > r \mid F_X(x) > r\right] \sim \mathcal{L}(1-r)(1-r)^{\frac{1}{\eta}-1} \quad \text{as } r \to 1$$

- Asymptotic Dependence (AD): $\chi > 0$ and $\eta = 1$
- Asymptotic Independence (AI): $\chi = 0$ and $\eta \neq 1$

Extremal Dependence Properties

Depending on the weighting function used, c_b has an influence in χ in some cases:

- If $\pi(u^*, v^*; \theta) = (u^* v^*)^\theta$ and c_t is AD, χ is dominated by χ_t with an influence of χ_b
- If $\pi(u^*, v^*; \theta) = (u^* v^*)^\theta$ and c_t is AI, χ is that from c_t
- If $\pi(u^*, v^*; \theta) = \exp\{-\theta(1 - u^*)(1 - v^*)\}$, χ is that from c_t (independently of the nature of c_t)

Extremal Dependence Properties

When c_b is a Frank copula (AI) with parameter $\beta \in \mathbb{R}$, c_t is a Gumbel copula (AD) with parameter $\alpha > 1$, and $\pi(u^*, v^*; \theta) = (u^* v^*)^\theta$, $\theta > 0$,

$$\chi = \frac{2 - 2^{1/\alpha}}{1 + \beta \left(1 - \exp\{-\beta\}\right)^{-1} \int_0^1 (1 - (v^*)^\theta) e^{-\beta(1-v^*)} \mathrm{d}v^*}$$

and $\eta = 1$

If $\pi(u^*, v^*; \theta) = \exp\{-\theta(1 - u^*)(1 - v^*)\}$,

$$\chi = 2 - 2^{1/\alpha} \quad \text{and} \quad \eta = 1$$

$(\chi_b = 0,\ \eta_b = 0.5,\ \chi_t = 2 - 2^{1/\alpha}$ and $\eta_t = 1)$

Extremal Dependence Properties

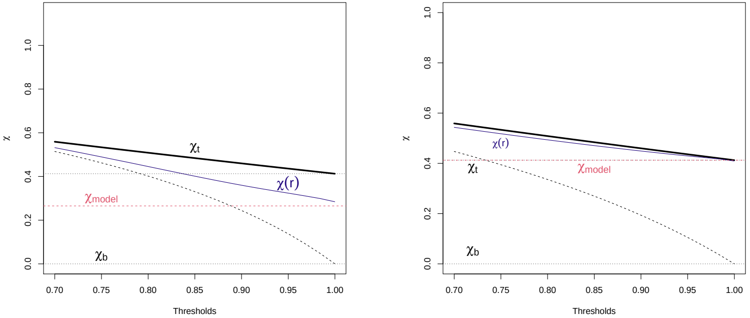


Figure 2: Weight functions: $\pi(u^*, v^*; \theta) = (u^* v^*)^\theta$ (left) and $\pi(u^*, v^*; \theta) = \exp\{-\theta(1-u^*)(1-v^*)\}$ (right) with $\gamma = (1.5, 2, 3.488889)$

Extremal Dependence Properties

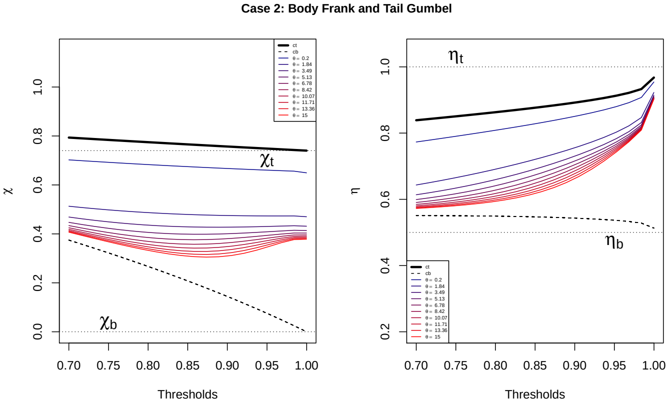


Figure 3: Weight function: $\pi(u^*, v^*; \theta) = (u^* v^*)^\theta$.

Extremal Dependence Properties

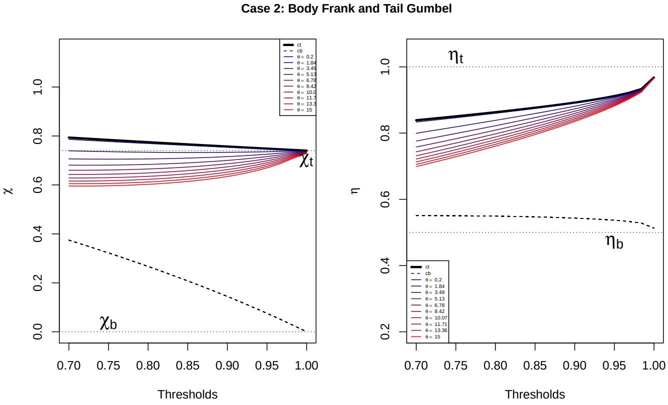


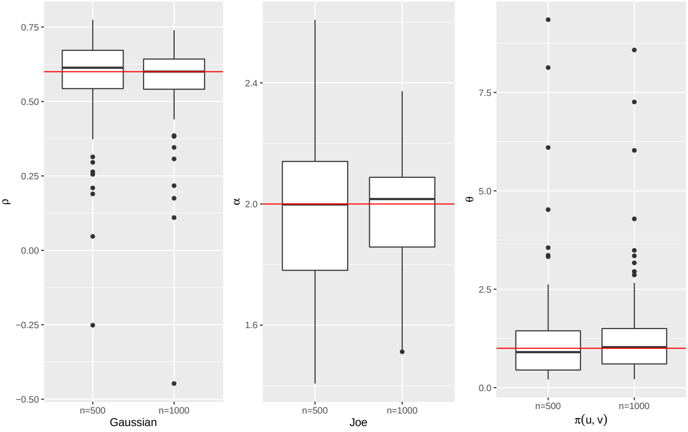
Figure 4: Weight function: $\pi(u^*, v^*; \theta) = \exp\{-\theta(1-u^*)(1-v^*)\}$.

Parameter Estimation

Simulation setup:

- c_t : Gaussian copula with $\rho = 0.6$
- c_b : Joe copula with $\alpha = 2$
- $\pi(u^*, v^*; \theta) = (u^* v^*)^\theta$ with $\theta = 1$
- $n = 500$ and $n = 1000$
- 100 repetitions

Parameter Estimation

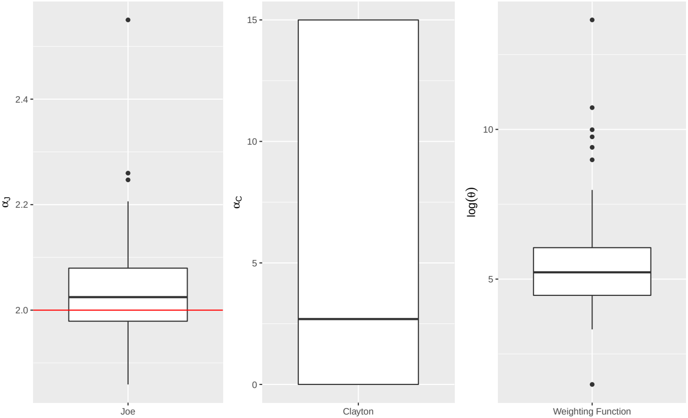


Model Misspecification

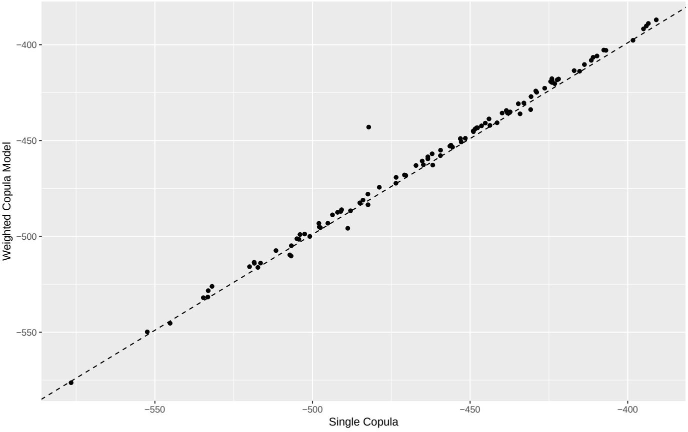
Simulation setup:

- True data from a Joe copula with $\alpha = 2$
- c_t : Clayton copula
- c_b : Joe copula (true)
- $\pi(u^*, v^*; \theta) = (u^* v^*)^\theta$
- $n = 1000$
- 100 repetitions

Model Misspecification



Model Misspecification

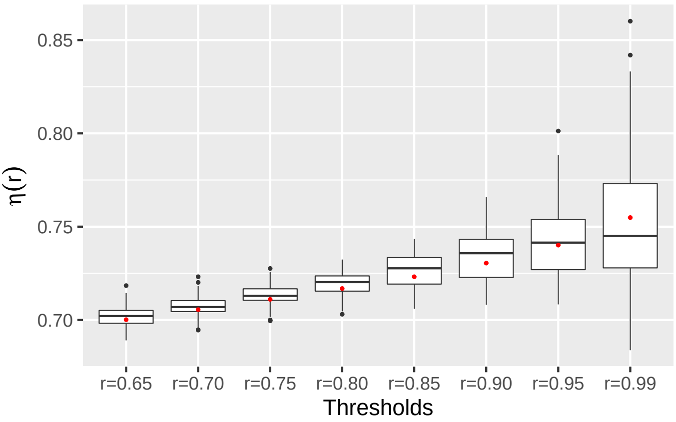


Model Misspecification

Simulation setup:

- True data from a Gaussian copula with $\rho = 0.65$
- Models considered:
  1. c_t : Joe copula; c_b : Frank copula
  2. c_t : Hüsler-Reiss copula; c_b : Clayton copula
  3. c_t : **Inverted Gumbel copula;** c_b : **Student t copula** $\rightarrow$ best average AIC
  4. c_t : Coles-Tawn copula; c_b : Galambos copula
- $\pi(u^*, v^*; \theta) = (u^* v^*)^\theta$
- $n = 1000$
- Each model was fitted 50 times

Model Misspecification



Ozone and Temperature Data

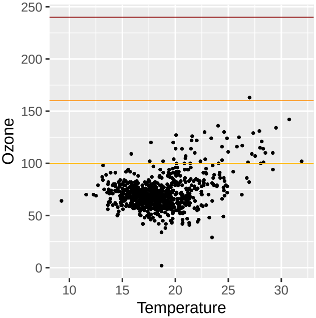
- Temperature may influence the levels of Ozone concentration in the air
- The legal thresholds for O_3 levels in the UK might then be found in the body and not just in the tails of the data

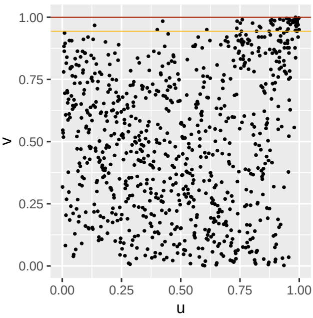
UK legal thresholds:

Levels	Low	Moderate	High	Very High
$O_3\,(\mu g/m^3)$	$[0, 100]$	$[101, 160]$	$[161, 240]$	> 240

We applied our model to the summers between 2011 and 2019 of Blackpool, UK

Ozone and Temperature Data



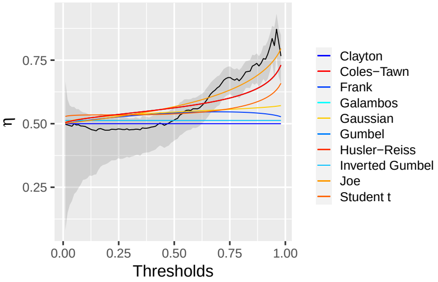


Apart from the upper tail, the variables seem to be negative correlated

Ozone and Temperature Data

Fitting a single copula

Copula	AIC
Clayton	2.0
Gaussian	-28.6
Frank	-15.8
Joe	**-143.6**
Gumbel	-97.4
Student t	-52.8
Inverted Gumbel	0.1
Hüsler-Reiss	-99.1
Coles-Tawn	-99.0
Galambos	-95.9



None of the single copulas showed negative correlation

Ozone and Temperature Data

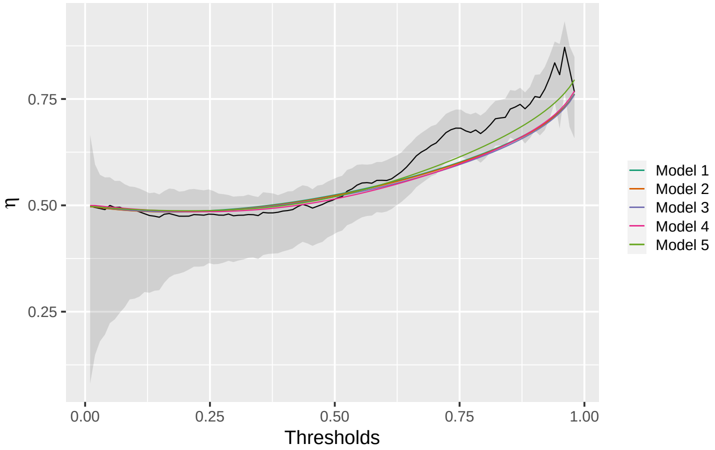
Fitting the weighted copula model with

$$\pi(u^*, v^*; \theta) = (u^* v^*)^{\theta}$$

Model		Parameters			AIC
c_b	c_t	$\hat{\beta}$	$\hat{\boldsymbol{\alpha}}$	$\hat{\theta}$	
Gaussian	Hüsler-Reiss	-0.40	1.24	0.35	-176.1
Gaussian	Galambos	-0.41	0.79	0.34	-172.1
Gaussian	Coles-Tawn	-0.33	0.35, 2.86	0.43	-158.4
Frank	Coles-Tawn	-2.52	0.33, 4.80	0.37	-163.2
Frank	Joe	-4.11	1.61	0.18	**-184.9**

The models with the best AIC all show negative correlation in the copulas tailored to the body

Ozone and Temperature Data



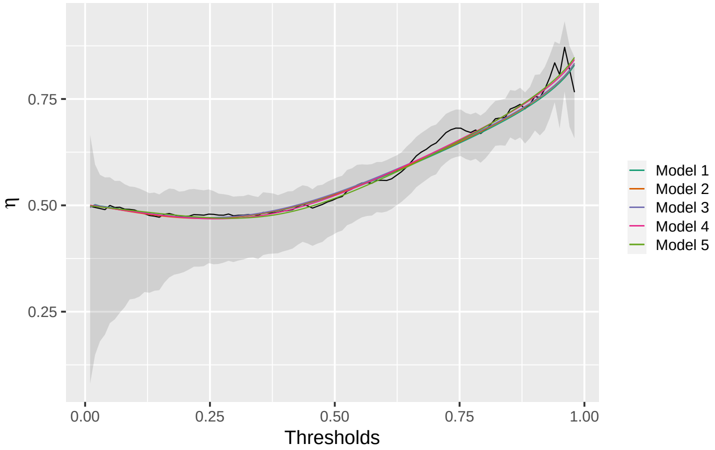
Ozone and Temperature Data

Fitting the weighted copula model with

$$\pi(u^*, v^*; \theta) = \exp\{-\theta(1-u^*)(1-v^*)\}$$

Model		Parameters			AIC
c_b	c_t	$\hat{\beta}$	$\hat{\boldsymbol{\alpha}}$	$\hat{\theta}$	
Gaussian	Hüsler-Reiss	-0.74	1.33	3.32	**-240.1**
Gaussian	Galambos	-0.72	0.90	3.55	-237.2
Gaussian	Coles-Tawn	-0.74	0.85, 0.79	3.25	-234.8
Frank	Coles-Tawn	-4.51	0.87, 1.02	4.33	-235.7
Frank	Joe	-6.49	1.72	2.45	-232.9

Ozone and Temperature Data



Ozone and Temperature Data

Other diagnostics

Models	Kendall's τ	$P[T \geq 24, O_3 \geq 100]$	$P[O_3 \geq 100 \mid 22 \leq T \leq 23]$
Empirical	0.0812	0.0302	0.1330
(95% CI)	(0.0173 , 0.1867)	(0.0147 , 0.0544)	(0.0227 , 0.1944)
Model 1	0.0690	0.0246	0.1441
Model 2	0.0663	0.0250	0.1412
Model 3	0.0770	0.0251	0.1429
Model 4	0.0779	0.0262	0.1392
Model 5	0.0718	0.0267	0.1366

Conclusions

- Our model provides a better fit than just fitting a single copula to the data
- It is flexible - it is able to capture different structures within the same data set
- However, it is computationally expensive
- Further Steps:
  - Account for non-stationarity - incorporate covariates

Questions?

Thank you all for listening!

References I

André, L. M., Wadsworth, J. L., and O'Hagan, A. (2023). Joint modelling of the body and tail of bivariate data. *Computational Statistics & Data Analysis*.

Aulbach, S., Bayer, V., and Falk, M. (2012). A Multivariate Piecing-Together Approach with an Application to Operational Loss Data. *Bernoulli*, 18:455–475.

Scarrott, C. and MacDonald, A. (2012). A Review of Extreme Value Threshold Estimation and Uncertainty Quantification. *Revstat Statistical Journal*, 10:33–60.

χ when $\pi(u^*, v^*; \theta) = (u^* v^*)^\theta$

$$
\begin{aligned}
c_1 &= 2 - 2^{1/\alpha} = \chi_{Gumbel}, \\
c_2 &= (2^{1/\alpha} - 1 - C_\alpha)(\theta - 1), \\
c_3 &= \beta\theta \left(1 - \exp\{-\beta\}\right)^{-1}, \\
c_4 &= 1, \\
c_5 &= -\theta/2 + o\left((1-r)^2\right), \quad \text{as } r \to 1 \\
c_6 &= \beta \left(1 - \exp\{-\beta\}\right)^{-1} \int_0^1 (1 - (v^*)^\theta) e^{-\beta(1-v^*)} \mathrm{d}v^*, \\
c_7 &= -\frac{1}{2} \int_0^1 B_{v^*,\beta,\theta} \mathrm{d}v^*
\end{aligned}
$$

with

$$
\begin{aligned}
B_{v^*,\beta,\theta} = {} & \frac{2\beta^2 (1 - (v^*)^\theta)(1 - \exp\{-\beta v^*\}) \exp\{-2\beta(1 - v^*)\}}{(1 - \exp\{-\beta\})^2} \\
& - \frac{\beta\theta (v^*)^\theta \exp\{-\beta(1 - v^*)\}}{1 - \exp\{-\beta\}} - \frac{\beta^2 (1 - (v^*)^\theta) \exp\{-\beta(1 - v^*)\}}{1 - \exp\{-\beta\}}
\end{aligned}
$$

χ when $\pi(u^*, v^*; \theta) = (u^* v^*)^\theta$

$$
\begin{aligned}
\chi &= \lim_{r \to 1} P\left[F_Y(y) > r \mid F_X(x) > r\right] \\
&= \lim_{r \to 1} \frac{c_1(1-r) + c_2(1-r)^2 + c_3(1-r)^3 + o\left((1-r)^3\right)}{c_4(1-r) + c_5(1-r)^2 + c_6(1-r) + c_7(1-r)^2 + o\left((1-r)^2\right)} \\
&= \lim_{r \to 1} \left(\frac{c_1}{c_4 + c_6} + \left[\frac{c_2 - c_1(c_5 + c_7)}{(c_4 + c_6)^2} \right] (1-r) + \mathcal{O}\left((1-r)^2\right) \right) \\
&= \frac{c_1}{c_4 + c_6} = \frac{2 - 2^{1/\alpha}}{1 + \beta\left(1 - \exp\{-\beta\}\right)^{-1} \int_0^1 (1 - (v^*)^\theta) e^{-\beta(1 - v^*)} \mathrm{d}v^*}
\end{aligned}
$$

χ when $\pi(u^*, v^*; \theta) = \exp\{-\theta(1-u^*)(1-v^*)\}$

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\begin{aligned}
c_1 &= 2 - 2^{1/\alpha} = \chi_{Gumbel}, \\
c_2 &= 1, \\
c_3 &= \frac{1}{\alpha}, \\
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\end{aligned}
$$

with

$$
\begin{aligned}
A_{v^*,\beta,\theta} = &- \frac{2\beta^2(1-\exp\{-\beta\})\exp\{-2\beta(1-v^*)\}}{(1-\exp\{-\beta\})^2} - \frac{\beta\theta(1-v^*)\exp\{-\beta(1-v^*)\}}{1-\exp\{-\beta\}} \\
&+ \frac{\beta^2\exp\{-\beta(1-v^*)\}}{1-\exp\{-\beta\}}
\end{aligned}
$$

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$$
\begin{aligned}
\chi &= \lim_{r\to 1} P\left[F_Y(y) > r \mid F_X(x) > r\right] \\
&= \lim_{r\to 1} \frac{c_1(1-r) + o\left((1-r)^2\right)}{c_2(1-r) + c_3(1-r)^2 + c_4(1-r)^2 + o\left((1-r)^2\right)} \\
&= \lim_{r\to 1} \left(\frac{c_1}{c_2} - \frac{c_3 + c_4}{c_2^2}(1-r) + \mathcal{O}\left((1-r)^2\right)\right) = \frac{c_1}{c_2} = 2 - 2^{1/\alpha}
\end{aligned}
$$