Lecture 23

Transport characteristics of electrons in a metal and electron gases in 2D semiconductor structures, scattering time and momentum relaxation rate.

Drude formula, diffusion coefficient, Einstein relation; diffusion equation and its solution.

Next time:

Interference of waves in disordered media, the phenomenon of enhanced backscattering.

Weak and strong localisation of electrons in disordered conductors.

Electrons in metals and semiconductors

$$\varepsilon(\vec{p}) \approx \frac{p^2}{2m} \Rightarrow \vec{v} = \frac{\vec{p}}{m}$$

the effective mass approximation

$$\vec{v} = \vec{\partial}_p \varepsilon(\vec{p})$$

group velocity

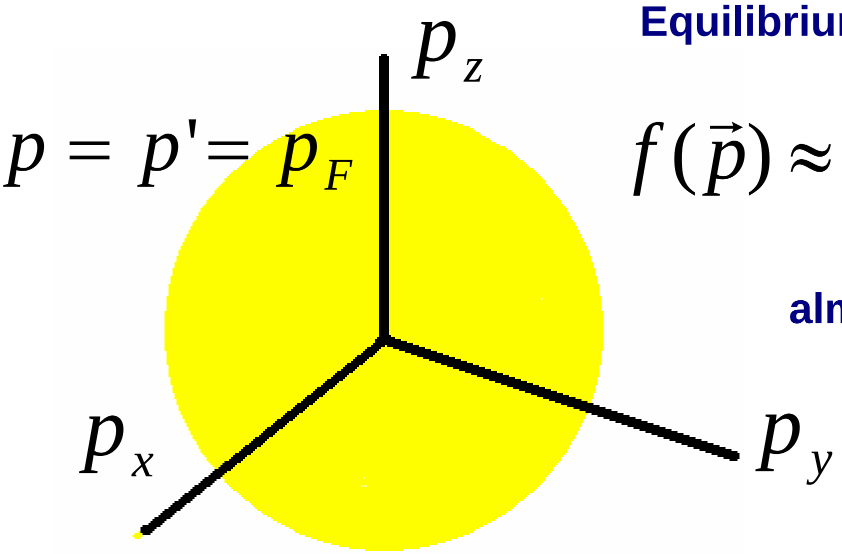
distribution function $f(t,\vec{r},\vec{p})$

probability density to find electron with momentum *p* at the coordinate *r*

$$n_e(t,\vec{r}) = \int \frac{d\vec{p}}{(2\pi\hbar)^d} \cdot f(t,\vec{r},\vec{p}) = \int d\varepsilon\, \gamma(\varepsilon) \cdot \bar{f}(t,\vec{r},\varepsilon)$$

electron density **density of states**

$$\bar{f} = \int \frac{dS_{\vec{p}}}{S_{\vec{p}}} \cdot f(t,\vec{r},\vec{p})$$



Equilibrium Fermi distribution function

$$f(\vec{p}) \approx f_T(\varepsilon) = \frac{1}{e^{[\varepsilon(\vec{p})-\varepsilon_F]/T} + 1}$$

almost a step function at $\varepsilon = \varepsilon_F$

$$eV, T << \varepsilon_F$$

Useful formulae for quick estimates

Density of electrons

$$n_e \sim \lambda_F^{-d} = \left(\frac{p_F}{2\pi\hbar} \right)^d$$

Density of states for a *d*-dimensional metal

$$\gamma_F \equiv \gamma(\varepsilon_F) = \int \frac{d\vec{p}}{(2\pi\hbar)^d} \delta(\varepsilon(\vec{p}) - \varepsilon_F)$$

$$\gamma_F \sim \frac{\lambda_F^{-d}}{\varepsilon_F} \sim \frac{n_e}{\varepsilon_F} \sim \frac{m n_e^{1-2/d}}{\hbar^2}$$

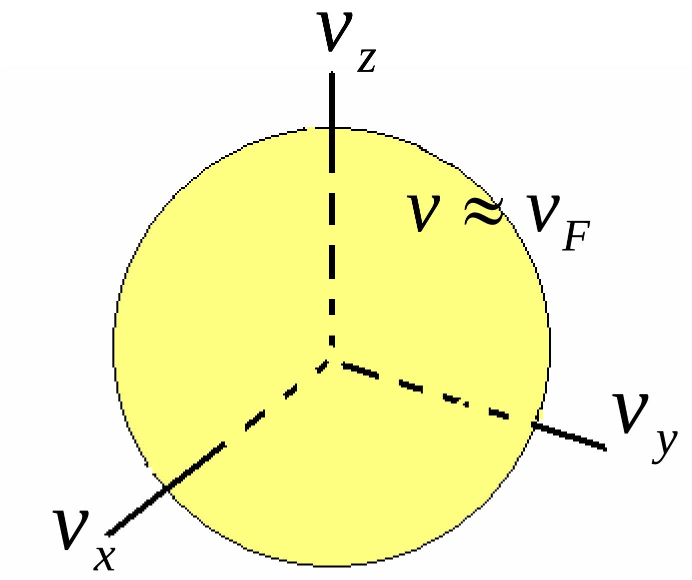
Fermi energy, Fermi wavelength and the density

$$\varepsilon_F = \frac{p_F^2}{2m} \sim \frac{\hbar^2}{m\lambda_F^2} \sim \frac{\hbar^2 \; n_e^{2/d}}{m}$$

Terminology: linear response regime

$$eV << T << \varepsilon_F$$

Electron current in a metal



$$\vec{j} = \int \frac{d\vec{p}}{(2\pi\hbar)^d} \; \vec{v}\, f(\vec{p})$$

Example: isotropic distribution $f(\vec{p}) = f(|\vec{p}|)$

such as $f_T(p)$

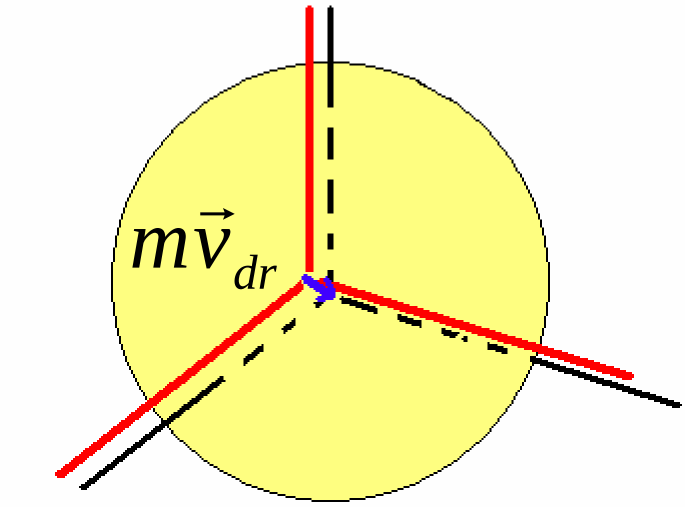
or, at least, inversion-symmetric $f(\vec{p}) = f(-\vec{p})$

$$\vec{j} = \int \frac{d\vec{p}}{(2\pi\hbar)^d} \; \vec{v}\, f(\vec{p}) \Big|_{\vec{p} \to -\vec{p}}$$

$$= \int \frac{d\vec{p}}{(2\pi\hbar)^d} \; (-\vec{v})\, f(-\vec{p})$$

$$= \int \frac{d\vec{p}}{(2\pi\hbar)^d} \; (-\vec{v})\, f(\vec{p}) = 0$$

Phenomenological approach to transport



Drift velocity $\vec{v}_{dr}$ describes the motion of the centre of mass of all electrons together:

$$\vec{v}_{dr} = \frac{d\vec{R}}{dt} \qquad \vec{R} = \frac{1}{N}\sum_j \vec{r}_j$$

$$f(\vec{p}) = f_T(\vec{p} - m\vec{v}_{dr})$$

$$\vec{j} = \int \frac{d\vec{p}}{(2\pi\hbar)^d} \; \vec{v}\, f_T(\vec{p} - m\vec{v}_{dr})$$

$$= \int \frac{d(\vec{p} - m\vec{v}_{dr})}{(2\pi\hbar)^d} \; (\vec{v} - \vec{v}_{dr}) f_T(\vec{p} - m\vec{v}_{dr})$$

$$+ \vec{v}_{dr} \int \frac{d(\vec{p} - m\vec{v}_{dr})}{(2\pi\hbar)^d} \; f_T(\vec{p} - m\vec{v}_{dr})$$

$$\vec{j} = \vec{v}_{dr} \int \frac{d\vec{p}}{(2\pi\hbar)^d} \; f_T(\vec{p}) = \vec{v}_{dr} n_e$$

How must a deviation of the distribution function from the equilibrium form look like to produce a finite current value:

$$f(\vec{p}) - f_T(\vec{p}) = f_T(\vec{p} - m\vec{v}_{dr}) - f_T(\vec{p}) \approx -m\vec{v}_{dr} \frac{\partial f_T}{\partial \vec{p}}$$

$$= -m\vec{v}_{dr} \frac{\partial \varepsilon}{\partial \vec{p}} \frac{\partial f_T}{\partial \varepsilon} = -m\vec{v}_{dr} \cdot \vec{v} \frac{\partial f_T}{\partial \varepsilon} \approx m\vec{v}_{dr} \cdot \vec{v}\, \delta(\varepsilon - \varepsilon_F)$$

Ohm's law

$$\vec{j}_e = \sigma_0 \vec{E}$$

$$\frac{d\vec{v}_{dr}}{dt} = \frac{e}{m}\vec{E} - \frac{\vec{v}_{dr}}{\tau}$$

in a steady state:

$$\frac{e}{m}\vec{E} - \frac{\vec{v}_{dr}}{\tau} = 0$$

$$\vec{v}_{dr} = \frac{e\tau}{m}\vec{E}$$

Relaxation time, τ describes frictional looses of the total electron momentum.

$$\vec{j}_e = en_e\vec{v}_{dr} = en_e\frac{e\tau}{m}\vec{E} = \frac{e^2 n_e \tau}{m}\vec{E} = \sigma_0\vec{E}$$

conductivity (Drude formula)

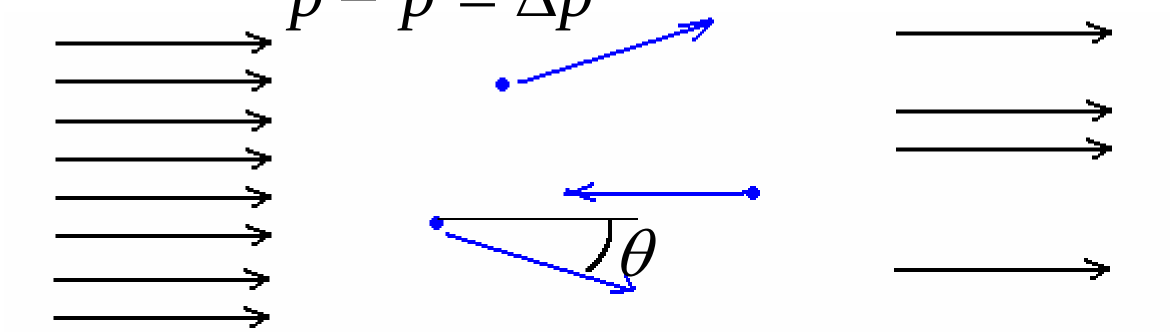
$$\boxed{\sigma_0 = \frac{e^2 n_e \tau}{m}}$$

resistivity

$$\rho_0 = \frac{1}{\sigma_0} = \frac{m}{e^2 n_e \tau}$$

Scattering rate and momentum relaxation rate

$$\vec{p} - \vec{p}\,' = \Delta\vec{p}$$



$$\Delta p_{beam} = p_F(1-\cos\theta)$$

$$\frac{v_{dr}}{\tau} = \frac{1}{mN}\frac{\Delta P}{\Delta t} = \frac{1}{m}\cdot\frac{mv_{dr}}{p_F}\cdot\int d\theta\, p_F(1-\cos\theta)w(\theta)$$

scattering rate to angle θ

Momentum relaxation rate

$$\tau^{-1} = \int (1-\cos\theta)w(\theta)d\theta$$

Determines so the called ‘transport time’

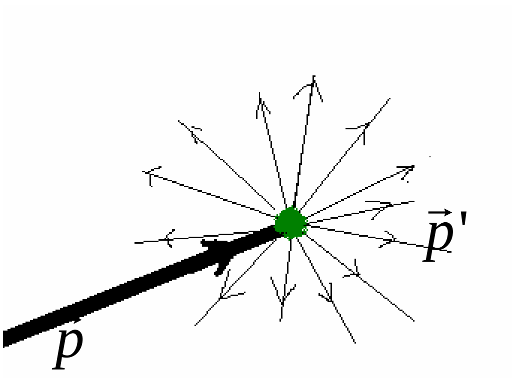
In the general situation, one has to distinguish momentum relaxation rate from the total scattering rate

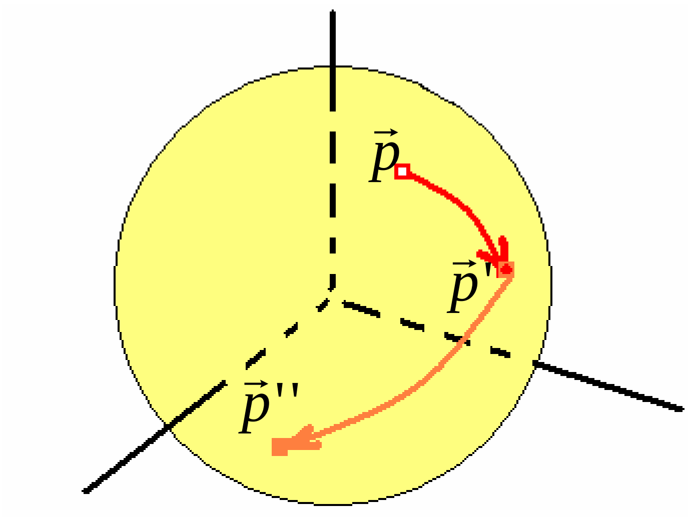
$$\tau_0^{-1} = \int w(\theta)d\theta$$

(the latter never enters the conductivity formulae)

Impurity scattering

$$H=\sum_i \frac{m}{2}\dot{\vec r}_i^{\,2}+\sum_{i\neq j}\frac{e^2/\chi}{2|\vec r_i-\vec r_j|}-\sum_i e\vec E\cdot\vec r_i+\underbrace{\sum_i^{n-imp} u(\vec r_i-\vec X_n)}_{\text{disorder}}$$





Impurity scattering may transfer the momentum taken from the electrons accelerated by an electric field to the lattice into which they are incorporated.

Born approximation for the impurity scattering:

$$w_{\vec p\to\vec p'}=\frac{2\pi}{\hbar}\cdot\left|\langle\vec p\,|\sum_n u(\vec r-\vec X_n)\,|\,\vec p'\rangle\right|^2\delta(\varepsilon(\vec p)-\varepsilon(\vec p'))$$

$$\left|\langle\vec p\,|\sum_n u(\vec r-\vec X_n)\,|\,\vec p'\rangle\right|^2=\left|\int d\vec r\,\frac{e^{i\vec r\cdot(\vec p-\vec p')/\hbar}}{L^d}\sum_n u(\vec r-\vec X_n)\right|^2$$

after averaging over X_n

$$=\sum_{n,m}\frac{e^{i(\vec X_n-\vec X_m)\cdot(\vec p-\vec p')}}{L^d}\left|\int d\vec r e^{i\vec r\cdot(\vec p-\vec p')}u(\vec r)\right|^2\approx\frac{N_{imp}}{L^d}\,|u_{\vec p-\vec p'}|^2=n_{imp}\,|u_{\vec p-\vec p'}|^2$$

Models for impurity scattering

$$w_{\vec{p}\to\vec{p}'} = n_{imp}\frac{2\pi}{\hbar}\cdot|u_{\vec{p}-\vec{p}'}|^2\,\delta(\varepsilon(\vec{p})-\varepsilon(\vec{p}'))$$

$$u_{\vec{p}-\vec{p}'} = \int d\vec{r}\cdot e^{i\vec{r}(\vec{p}-\vec{p}')/\hbar}u(\vec{r})$$

δ **- functional scatterer,** $u(\vec{r}) = u\cdot\delta(\vec{r})$

$$u_{\vec{p}-\vec{p}'} = \int d\vec{r}\cdot e^{i\vec{r}(\vec{p}-\vec{p}')/\hbar}u\cdot\delta(\vec{r}) = u$$

results in the isotropic scattering (independent of the angle θ**)**

$$\varepsilon(\vec{p}) = \varepsilon(\vec{p}') \approx \varepsilon_F \Rightarrow p = p' \approx p_F \qquad \cos\theta = \frac{\vec{p}\cdot\vec{p}'}{p_F^2}$$

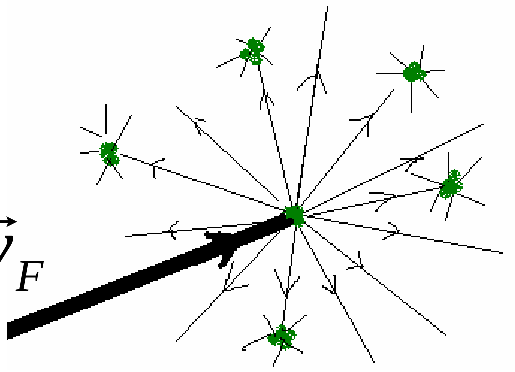
$$w(\theta) = n_{imp}\frac{2\pi}{\hbar}u^2\gamma_F \equiv w_0$$

For isotropic scattering

$$\tau^{-1} \equiv \int_0^{\pi}(1-\cos\theta)w_0\,d\theta = \int_0^{\pi}w_0\,d\theta \equiv \tau_0^{-1}$$

Electrons in a ‘dirty’ metal

$$\frac{\vec{p}_F}{m} = \vec{v}_F$$



momentum relaxation time

$$\downarrow$$

$$l = v_F \tau_1$$

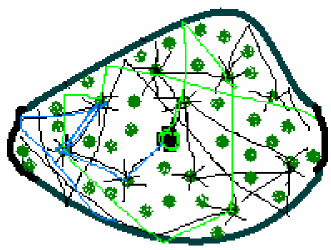
$$\uparrow$$

length of the mean free path

Electron propagation in a disordered metal is a random walk process.

Diffusion coefficient

$$D = \frac{v_F l}{d} = \frac{v_F^2 \tau}{d}$$



Diffusion equation

$$\partial_t \delta n_e = D \nabla^2 \delta n_e$$

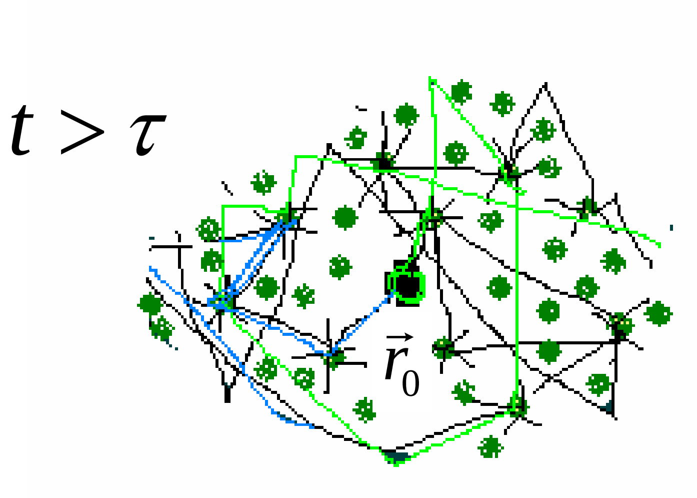
$$\delta n_e = \int d\varepsilon \gamma_F \bar{f}(t, \vec{r}, \varepsilon)$$

$$\vec{j}_{density} = -D \nabla \delta n_e$$

Conductivity of a disordered metal (Einstein’s formula)

$$\sigma_0 = e^2 \nu_F D = \frac{e^2 n_e \tau}{m}$$

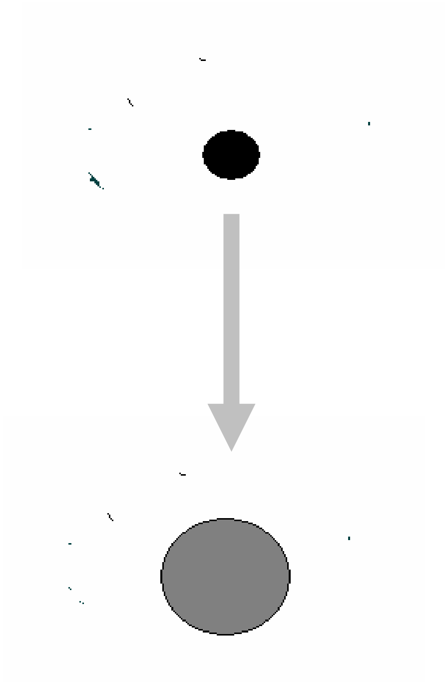
Gives the same as the Drude formula



Propagation of a multiply scattered electron along tree-like paths can be envisaged as a diffusive spreading across the sample of the probability density to find an electron.

$$[\partial_t - D\nabla^2]\ P(t;\vec{r},\vec{r}_0) = \delta(t)\cdot\delta(\vec{r}-\vec{r}_0)$$

Diffusion equation and its solution



$$P = \frac{\exp\{-(\vec{r}-\vec{r}_0)^2/Dt\}}{(Dt)^{d/2}}$$

$$P(t;\vec{r}_0,\vec{r}_0) = \frac{1}{(Dt)^{d/2}}$$

probability 'to return'

Old hunter’s wisdom: any rabbit will finally get into his trap

$$\int_\tau^t \upsilon dt\ P(t;\vec{r}_0,\vec{r}_0) = \int_\tau^t \frac{\upsilon dt}{(Dt)^{d/2}} = \frac{\upsilon}{D^{d/2}} \times \begin{cases} \ln\frac{t}{\tau} \to \infty & d=2 \\ \sqrt{t} \to \infty & d=1 \end{cases}$$

Space-time relations in diffusive systems

$$P(\vec{r},t) = \frac{e^{-\frac{(\vec{r}-\vec{r}_0)^2}{tD}}}{(Dt)^{d/2}}$$

typical distance from the injection point for a particle undergoing random walk

$$L = |\vec{r}-\vec{r}_0| \sim \sqrt{tD}$$

Macroscopic classical systems

$$L >> l >> \lambda_F \, (and \;\; \lambda_F \to 0)$$

electrons propagation is described using classical laws.

$$l \sim L_\varphi$$

Mesoscopic quantum systems

$$L \sim L_\varphi > l >> \lambda_F$$

Regime of multiply scattered phase-coherent <u>waves</u> which interference affects transport characteristics.

$$\boxed{\tau_\varphi >> \tau}$$

phase-coherence time and length

$$\boxed{L_\varphi = \sqrt{\tau_\varphi D} >> l}$$

Ballistic – ‘nanoscopic’

(discussed in the previous lectures)

$$L \sim \lambda_F << l$$