

Review Article:

S. Girvin Les Houches 1998

cond-mat/9907002

~120 pages

Selected Topics in the Physics of Quantum Hall Systems

John Chalker, Oxford University

Lecture One: Overview of the integer and fractional quantum Hall effects

Plateau transitions as quantum critical points

The Laughlin wavefunction and fractionally charged quasiparticles

Lecture Two: Broken symmetries in quantum Hall systems

Stripe and bubble phases

Quantum Hall ferromagnets and Skyrmions

Bilayers

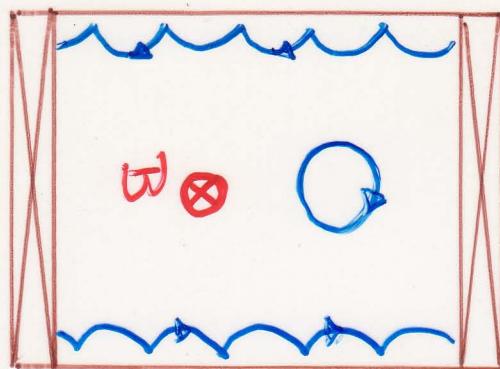
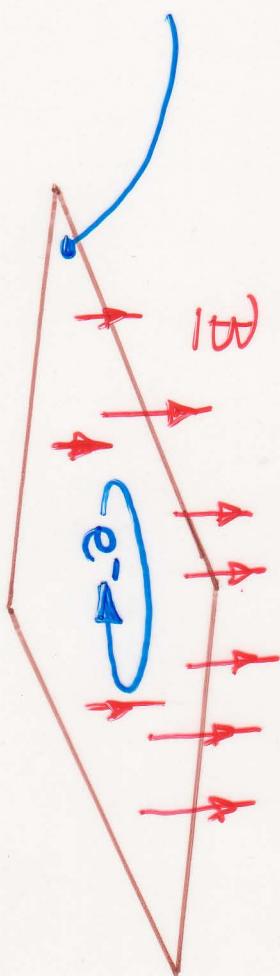
Seminar: Transport between coupled quantum Hall edge states in multilayer samples

Multilayer samples and the Quantum Hall effect in 3D

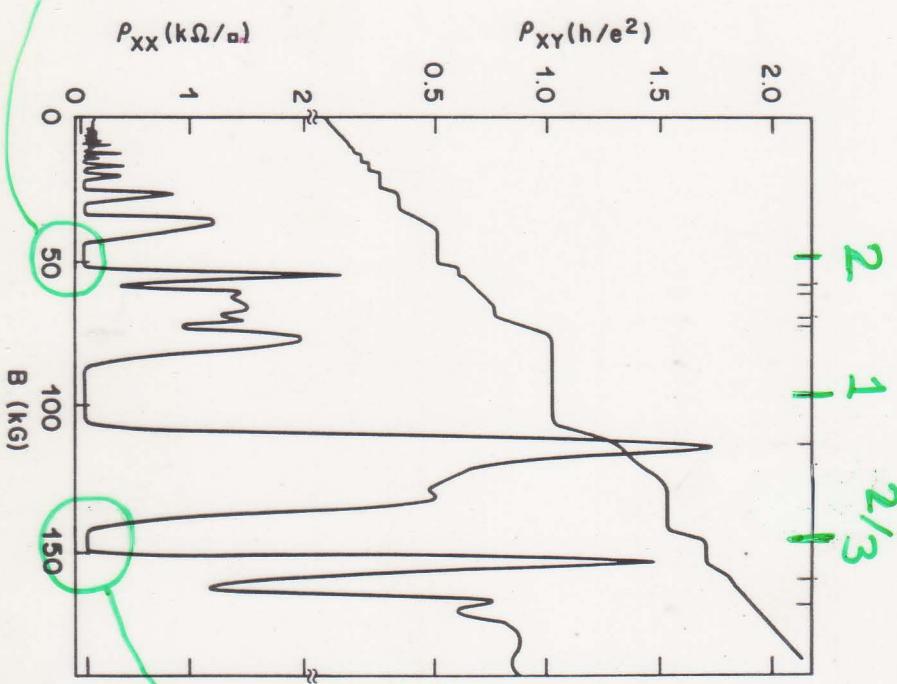
Interaction and disorder effects in coupled edge states

Aspects of the QHE

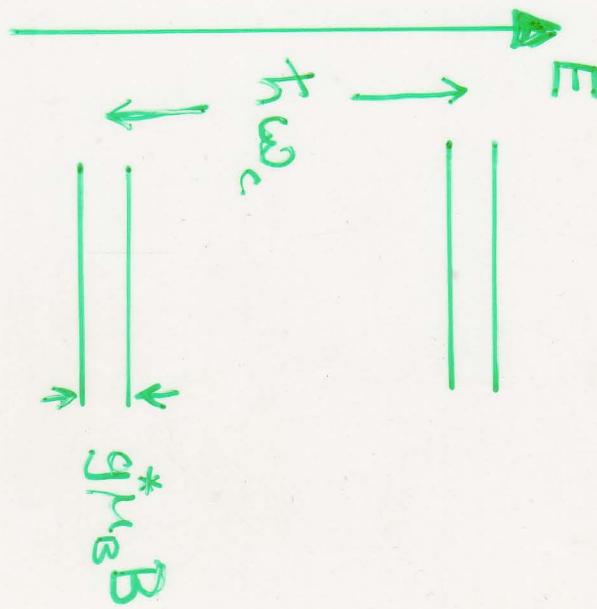
2D
electron
gas



Edge states
and
dissipationless
transport



Degeneracy
and
correlated states

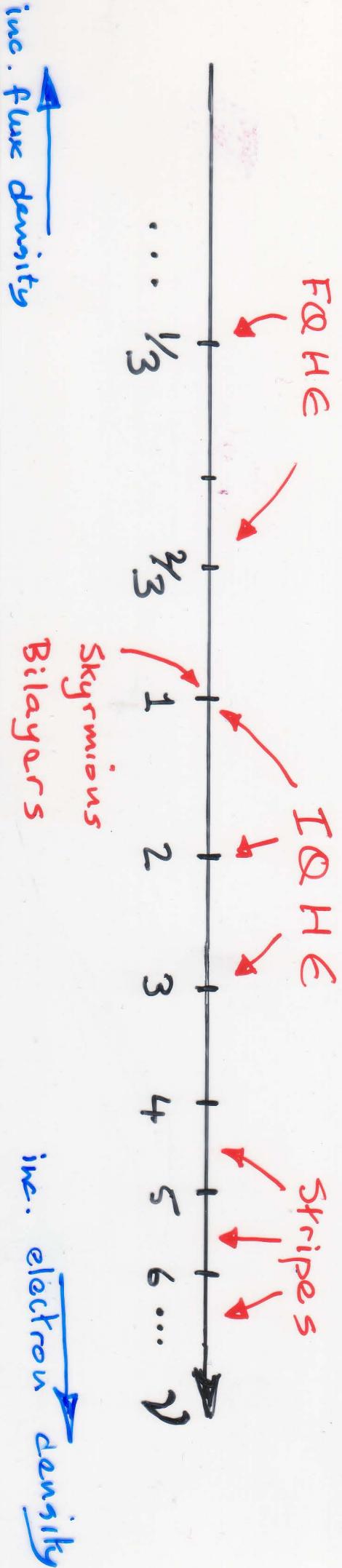


Overview of Phenomena

The control parameter

$$\text{filling factor } \nu = \frac{\# \text{ electrons}}{\# \text{ flux quanta} \sim \frac{B \times \text{Area}}{h/e}} n \times \text{Area}$$

free electrons: $\delta_{\text{Hall}} = \frac{ne}{B} = \frac{e^2}{h} \nu$



Essentials of QHE

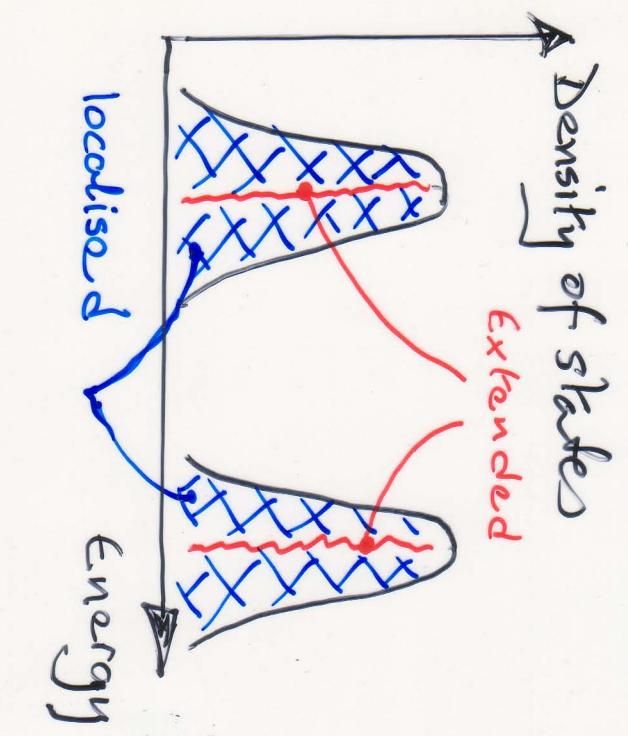
IQHE

(Magnetic field \rightarrow Energy gap)

Disorder \rightarrow localisation

FQHE

Interactions \rightarrow Energy gap
(Disorder \rightarrow localisation)



Laughlin states:

$$|\Psi(\xi_1 \dots \xi_N)|^2 \sim |\xi_i - \xi_j|^{2p}$$

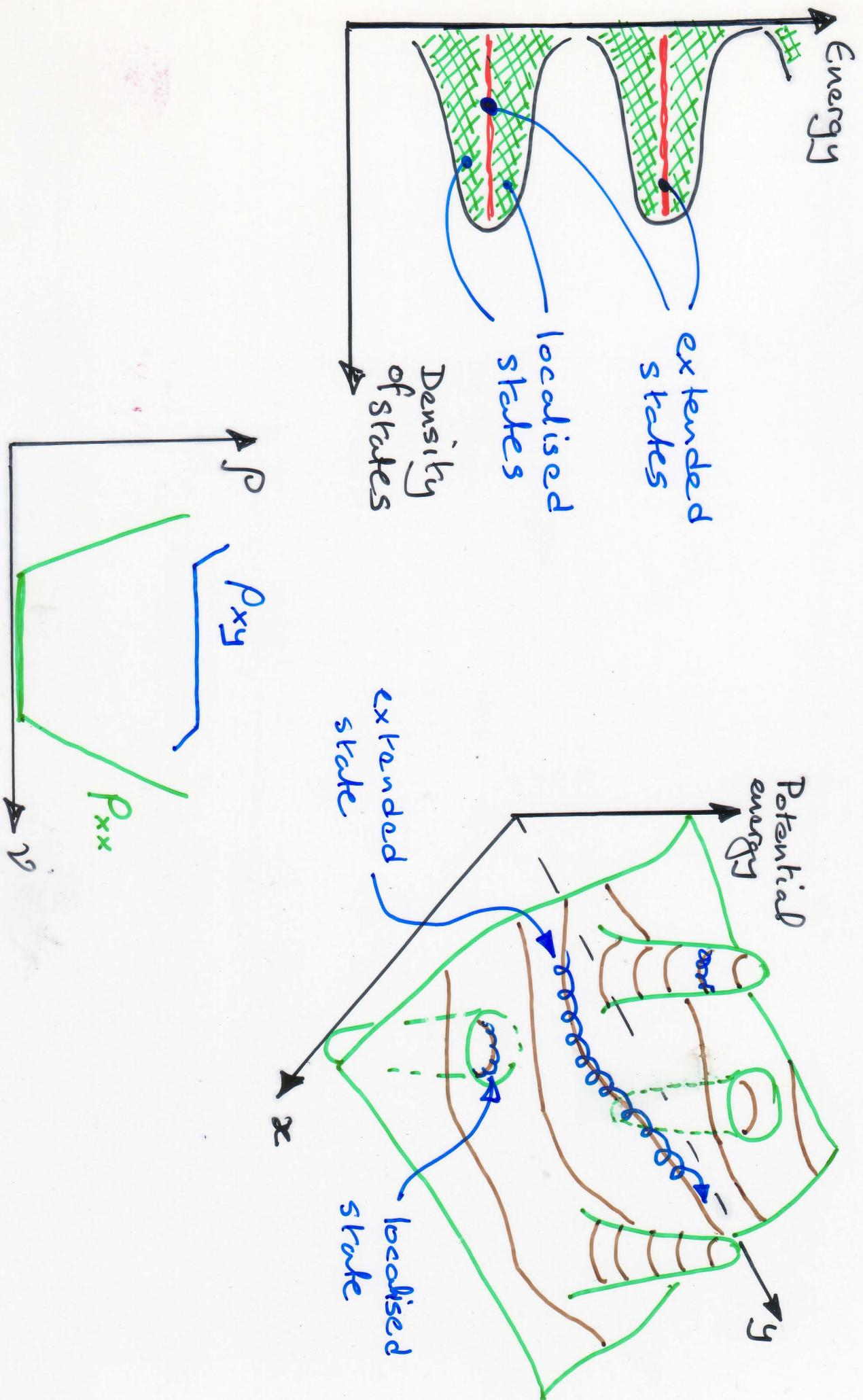
$p=3, 5 \dots$

$$\text{as } |\xi_i - \xi_j| \rightarrow 0$$

Electrons in localised states don't carry current

Correlations keep electrons apart

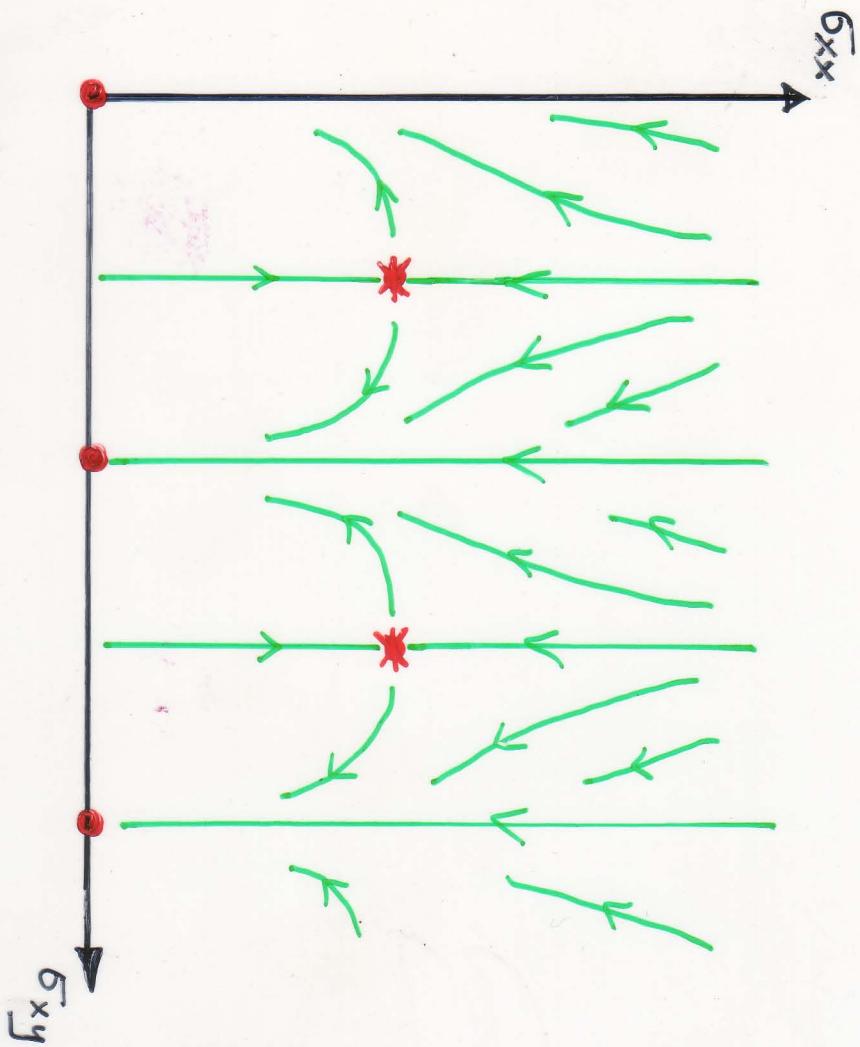
Why are there Hall plateaus?



TQHE Plateau Transition & Scaling

$\tilde{\sigma}$ -model with $\tilde{\sigma}_{xx}$ & $\tilde{\sigma}_{xy}$ as coupling constants

Scaling near critical Point



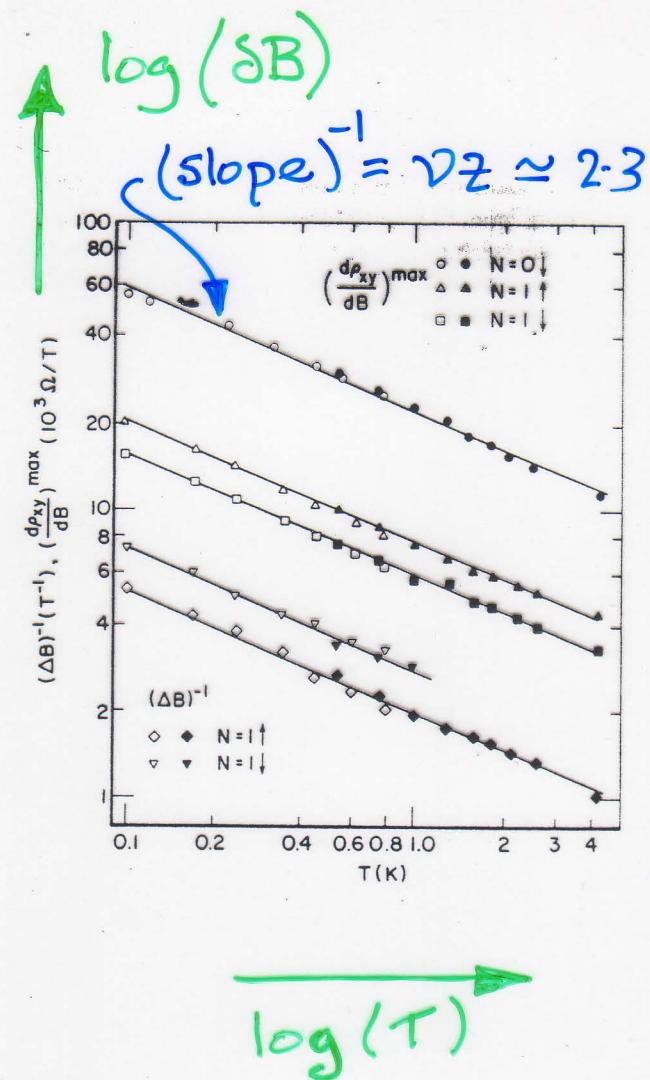
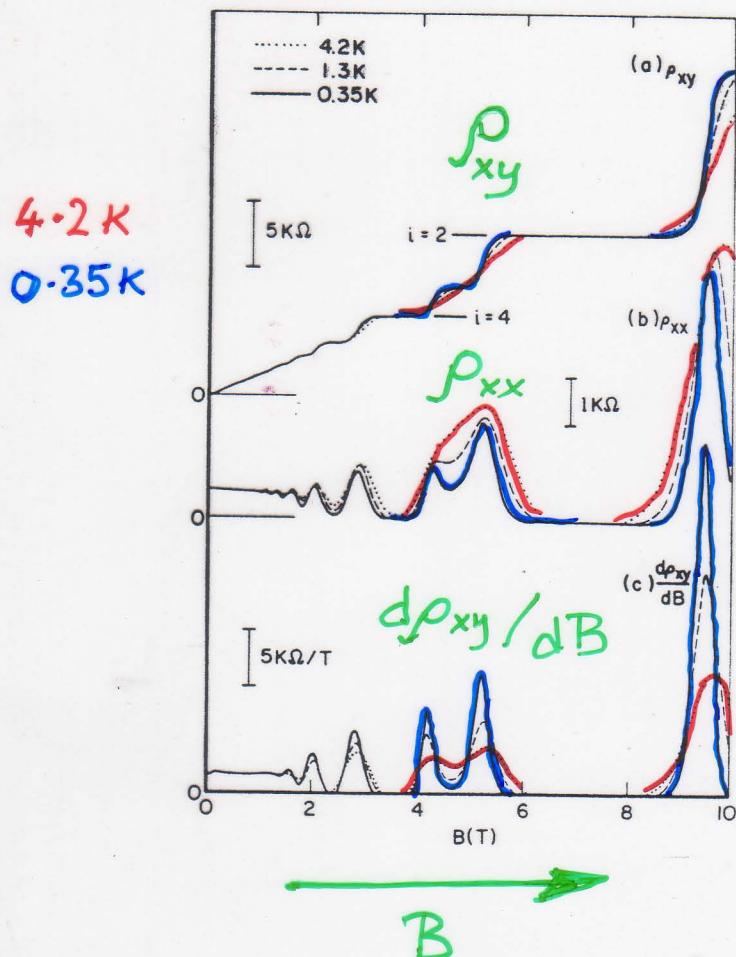
Correlation length: $\xi \sim |\Delta B|^{-\nu}$

Correlation time: $\tau \sim \xi^z$

Energy scale: $\frac{\hbar}{\tau} \sim |\Delta B|^{\nu z}$

$$\sigma = F\left(\left|\frac{\Delta B}{T}\right|^{\nu z}, \frac{C}{T}, \dots\right)$$

Experiments on Scaling



Wei et al PRL 61

Also scaling with

- frequency
- electric field
- sample size

See Haug et al
for recent results

Fractional Quantum Hall Effect

Central ideas

At special ν :

- Highly correlated ground state
- Energy gap for excitations
→ incompressibility
- Fractionally charged quasiparticles
 - treat using trial wavefunctions

Electrons in lowest Landau level

Single particle wavefunctions

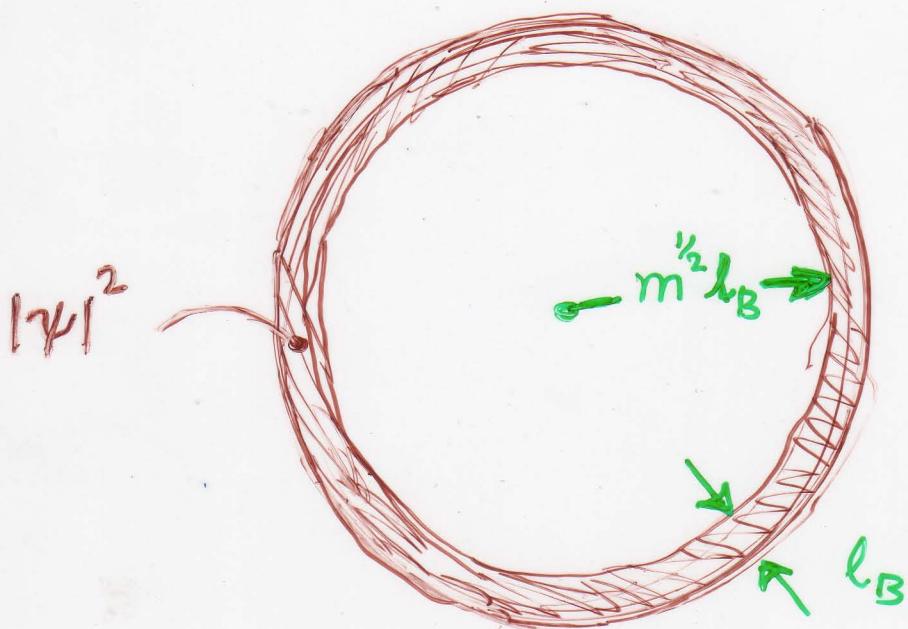
Circular gauge: $\vec{A} = \frac{B}{2}(y, -x, 0)$

with $l_B \equiv (\hbar/eB)^{1/2} = 1$

Complex coordinates: $z = x + iy$

$$\psi_m(x, y) \propto z^m e^{-|z|^2/4}$$

$$m = 0, 1, 2 \dots$$



Two particles in lowest Landau level

$$\psi(z_1, z_2) = f(z_1, z_2) e^{-\frac{1}{4}(|z_1|^2 + |z_2|^2)}$$

↑
polynomial in $z_1 + z_2$

Pauli $\Rightarrow f(z_1, z_2)$ antisymmetric

Forced to take:

$$f(z_1, z_2) = (z_1 - z_2)^l (z_1 + z_2)^m \quad \begin{matrix} \text{integer} \\ l, m \geq 0 \end{matrix}$$

↑
relative motion

↑
centre of mass motion

- Relative motion fixed by confinement to lowest Landau level
(incompressible state)

Pauli: l odd integer

Many particles : Laughlin wavefunction

Try variational form

$$\psi_L(z_1 \dots z_N) = \prod_{i < j} f(z_i - z_j) e^{-\frac{1}{4} \sum_k |z_k|^2}$$

N electrons correlated in pairs

Natural to pick $f(z_i - z_j)$ so as to minimise probability for electrons to get close together

In fact, have no freedom

- Lowest Landau level $\rightarrow f(z_i - z_j)$ is polynomial
 - Eigenstate of total angular momentum \rightarrow polynomial is homogeneous
 - Pauli $\rightarrow q$ is odd integer
- $$f(z_i - z_j) \propto (z_i - z_j)^q$$

Making quasi particles

Add hole at ξ

$$\psi(z_1 \dots z_N) = \prod_e (z_e - \xi) \times \psi_L$$

↑
electrons
avoid ξ

↑
Laughlin
wavefunction

Understanding of $\psi(z_1, z_2 \dots z_N)$

- the plasma analogy

Hard problem to find properties of state ψ

Use analogy with classical statistical mechanics

Suppose

$$|\psi(z_1, \dots z_N)|^2 = \exp(-\beta H_{cl}(z_1, \dots z_N))$$

- defines $H_{cl}(z_1, \dots z_N)$

Wavefunction

$$\psi(z_1, \dots z_N) = \prod_e (z_e - \xi) \prod_{i < j} (z_i - z_j)^q e^{-\frac{1}{4} \sum_k |z_k|^2}$$

Equivalent classical system

$$H_{cl}(z_1, \dots z_N) = -\frac{2q}{\beta} \sum_{i < j} \ln |z_i - z_j| + \frac{1}{2\beta} \sum_k |z_k|^2$$

$$= \frac{2}{\beta} \sum_e \ln |z_e - \xi|$$

Compare with electrostatics in 2D

Point charge Q at origin

$$\text{Electric field } E(r) = \frac{Q}{2\pi\epsilon_0 r}$$

$$\text{Electrostatic potential } V(r) = -\frac{Q}{2\pi\epsilon_0} \ln(r)$$

Compare with Hcl

$$\text{Set } \frac{1}{\beta} = \frac{q}{4\pi\epsilon_0}$$

H_{cl} is energy of classical charges q :

- Interacting with each other:

$$- \frac{q^2}{2\pi\epsilon_0} \sum_{i < j} \ln |z_i - z_j|$$

- Interacting with unit charge at ξ :

$$- \frac{q}{2\pi\epsilon_0} \sum_e \ln |z_e - \xi|$$

- Interacting with potential from background charge of opposite sign

$$+ \frac{q}{8\pi\epsilon_0} \sum_k |z_k|^2$$

Background
charge
density?

$$\rho = \nabla^2 \left(\frac{|z|^2}{8\pi} \right) = \frac{1}{2\pi}$$

Screening in classical plasmas

and

Properties of Laughlin wavefunction

Principle

Charge fluctuations cost energy

- Electron density matches background

Background charge density $\frac{1}{2\pi}$

Elections = charges q

so number density $\frac{1}{2\pi q}$

- Depletion of electron density at quasiparticles

"Hole" represented by unit classical charge

Screened by $\frac{1}{q}$ of an electron

Quasiparticles have
fractional charge

Direct observation of a fractional charge

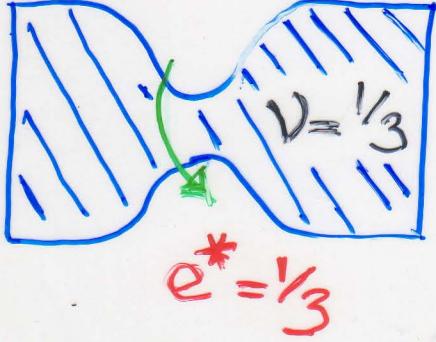
R. de-Picciotto, M. Reznikov, M. Heiblum, V. Umansky,
G. Bunin & D. Mahalu

Braun Center for Submicron Research, Department of Condensed Matter Physics,
Weizmann Institute of Science, Rehovot 76100, Israel

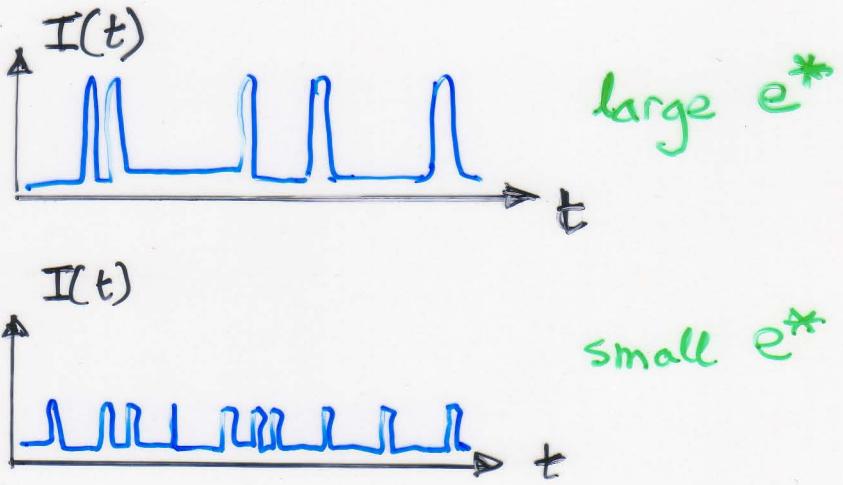
Nature 1997

Measure
shot noise
power

vs
 $\langle I \rangle$



Noise
Power

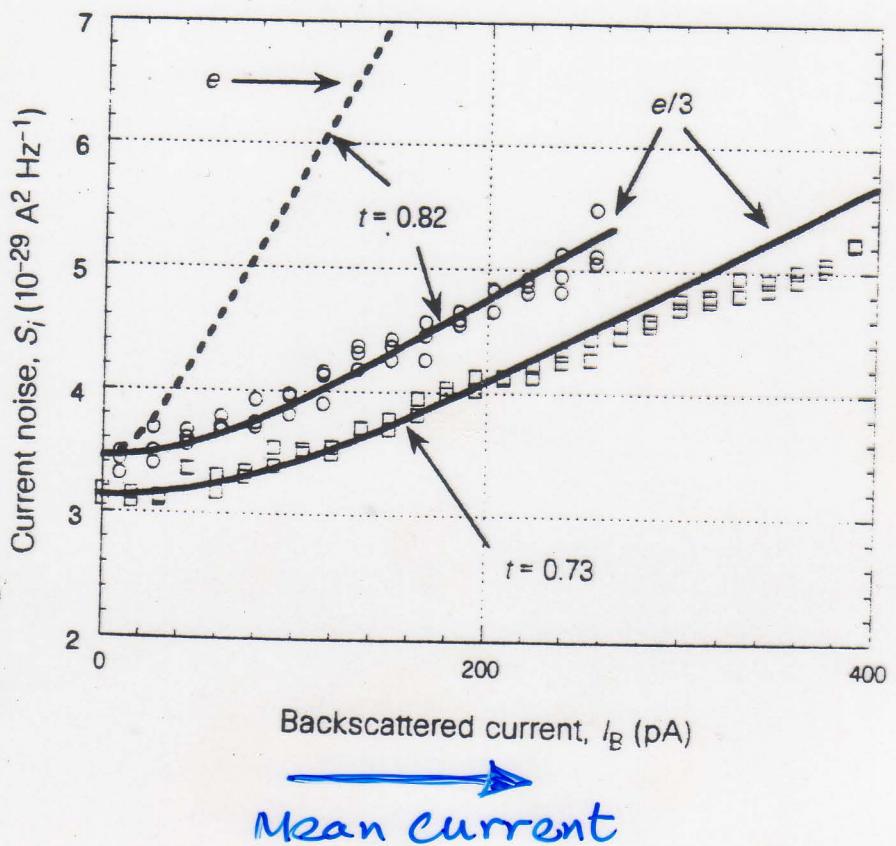


Expect

$$\langle S \rangle = 2e^* \langle I \rangle$$

t
noise power

- measure e^*



Lecture I: Summary

Integer quantum Hall effect

- plateau transition as quantum phase transition

Fractional quantum Hall effect:

- Laughlin wavefunction and fractionally-charged quasiparticles

Omissions:

Neutral excitations

Composite fermions

$\nu = 1/2$

Topological order

Fractional quantum Hall edge states ...

Lecture II: Broken symmetries in quantum Hall systems

Stripe and bubble phases

What takes the place of the FQHE in high Landau levels?

Quantum Hall ferromagnets and skyrmions

Electron spin as a degree of freedom

Charged quasiparticles with large spin

Bilayers

Spontaneous interlayer phase coherence

Excitonic superfluidity

Quantum Hall Systems at weak field

Energy Scales

Cyclotron energy $\hbar\omega_c = \frac{eB}{m^*}$

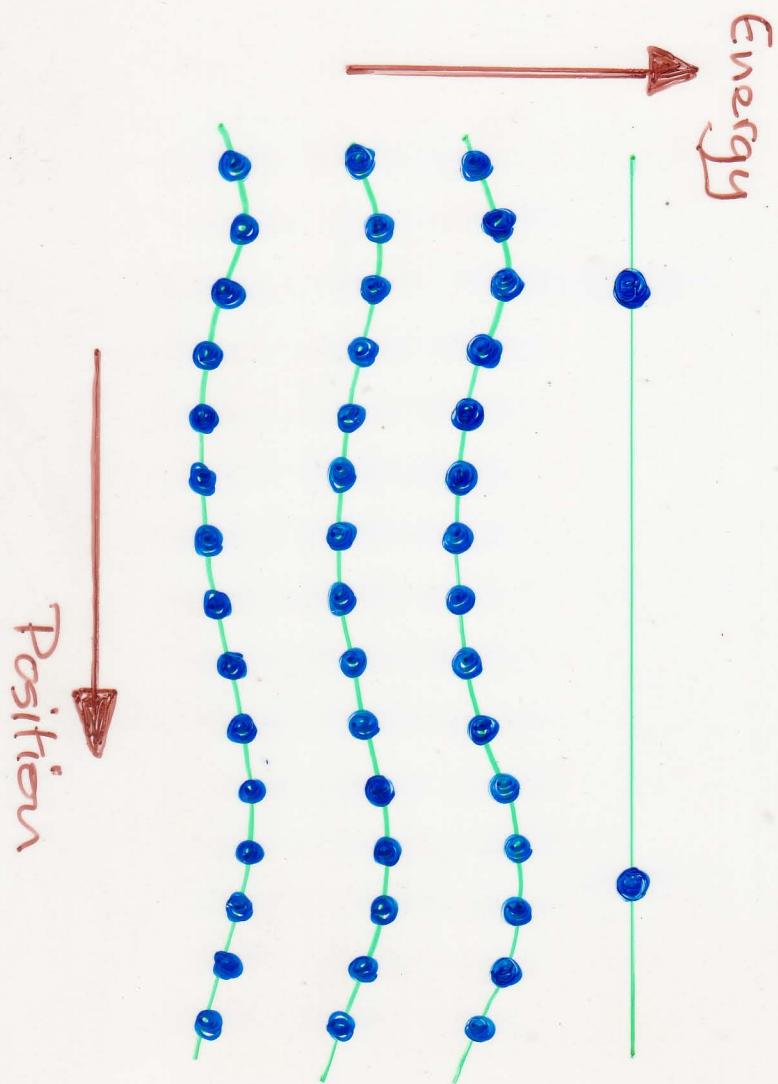
Coulomb energy $\frac{e^2}{4\pi\epsilon_0 l_B} \propto B^{1/2}$

Length Scales

Magnetic length $\lambda_B = \left(\frac{\hbar}{eB}\right)^{1/2}$

Larmor radius $R_c = \sqrt{V} \cdot l_B$

Screening by polarisation of filled Landau levels



Why is FQHE a strong-field phenomenon?

Length scales

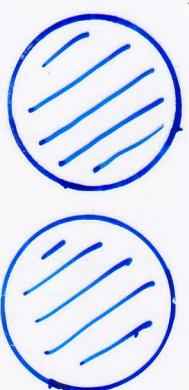
Two neighbouring electrons

Magnetic length

$$\ell_B = \left(\frac{e\hbar}{eB} \right)^{1/2}$$

- electron-electron separation within Landau level

In lowest Landau level

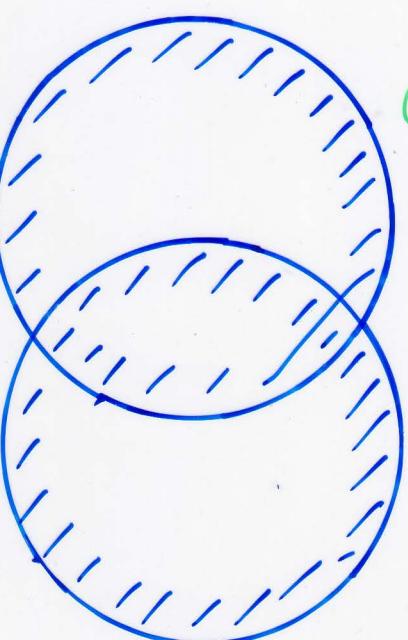


$$2\ell_B$$

Larmor radius

$$R_c = \sqrt{\nu} \ell_B$$

In higher Landau level

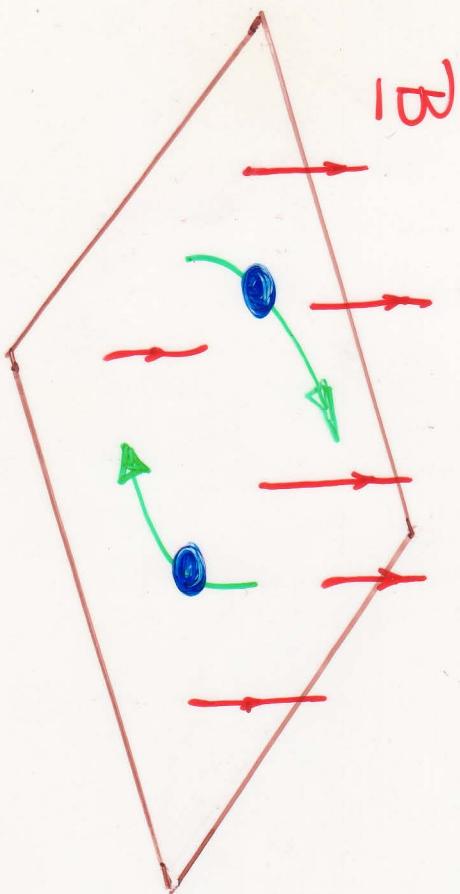


$$2R_c$$

- radius of cyclotron orbit

Correlated States at Strong and Weak Field

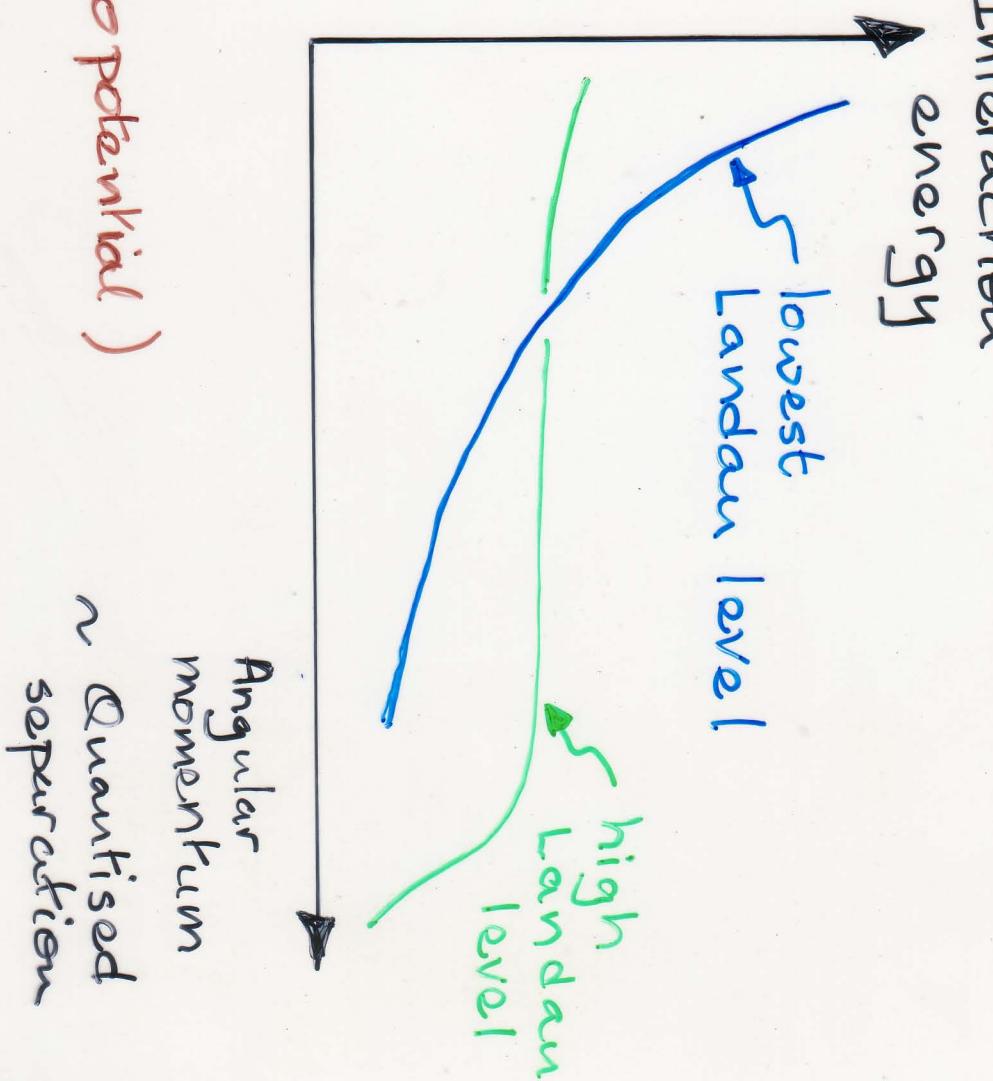
Two-particle states



Interaction energy

lowest Landau level

high Landau level



(Haldane Pseudo potential)

Two-dimensional electron gas in a strong magnetic field

H. Fukuyama

The Institute for Solid State Physics, The University of Tokyo, Tokyo, Japan

P. M. Platzman and P. W. Anderson*

Bell Laboratories, Murray Hill, New Jersey 07974

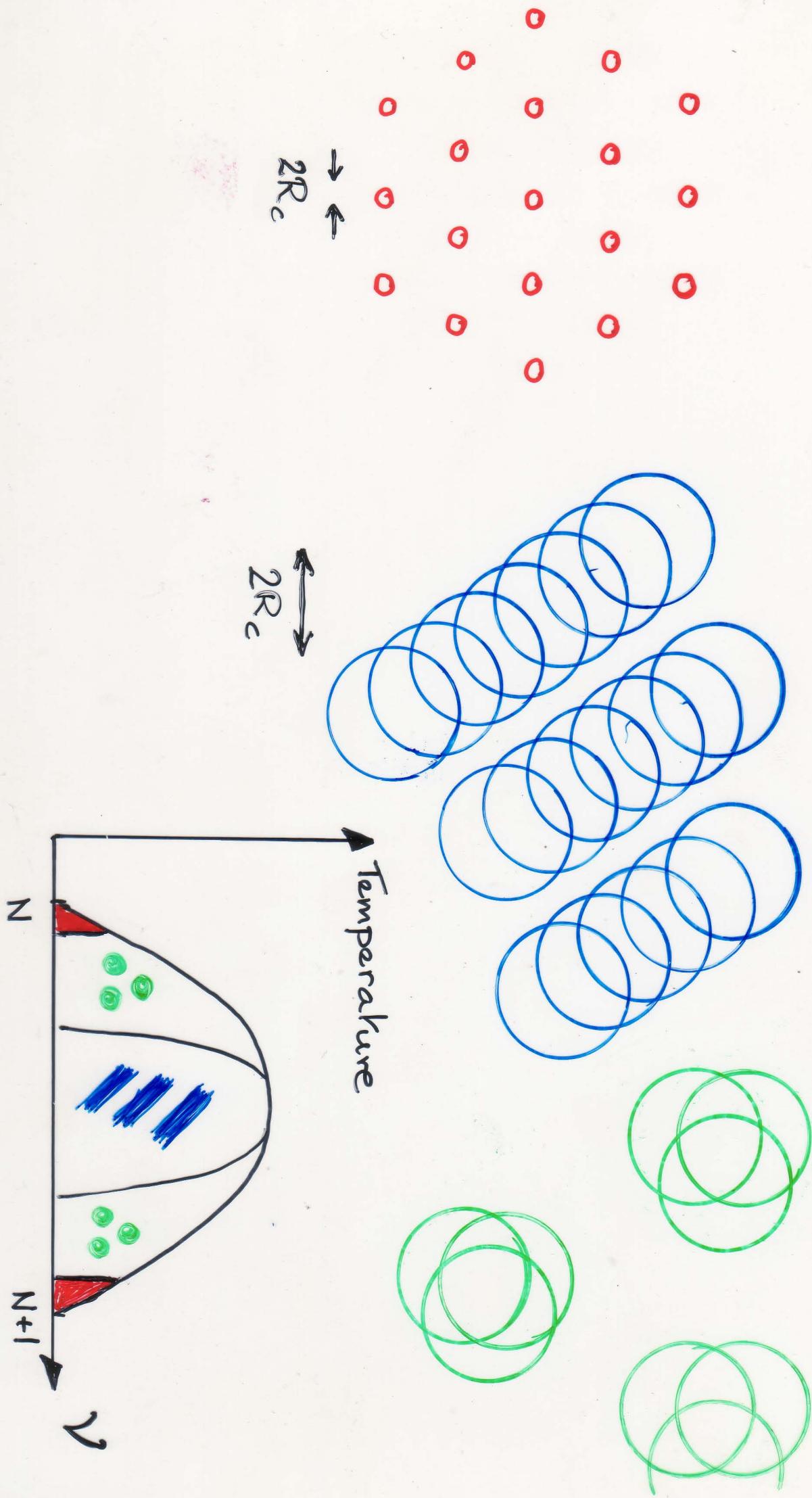
(Received 15 May 1978; revised manuscript received 29 December 1978)

Some interesting properties of the phase diagram of a two-dimensional electron gas are calculated within the framework of Hartree-Fock picture. We find that the system is unstable to the formation of a charge-density wave at temperatures well above the classical Wigner solid transition temperature.

I. INTRODUCTION

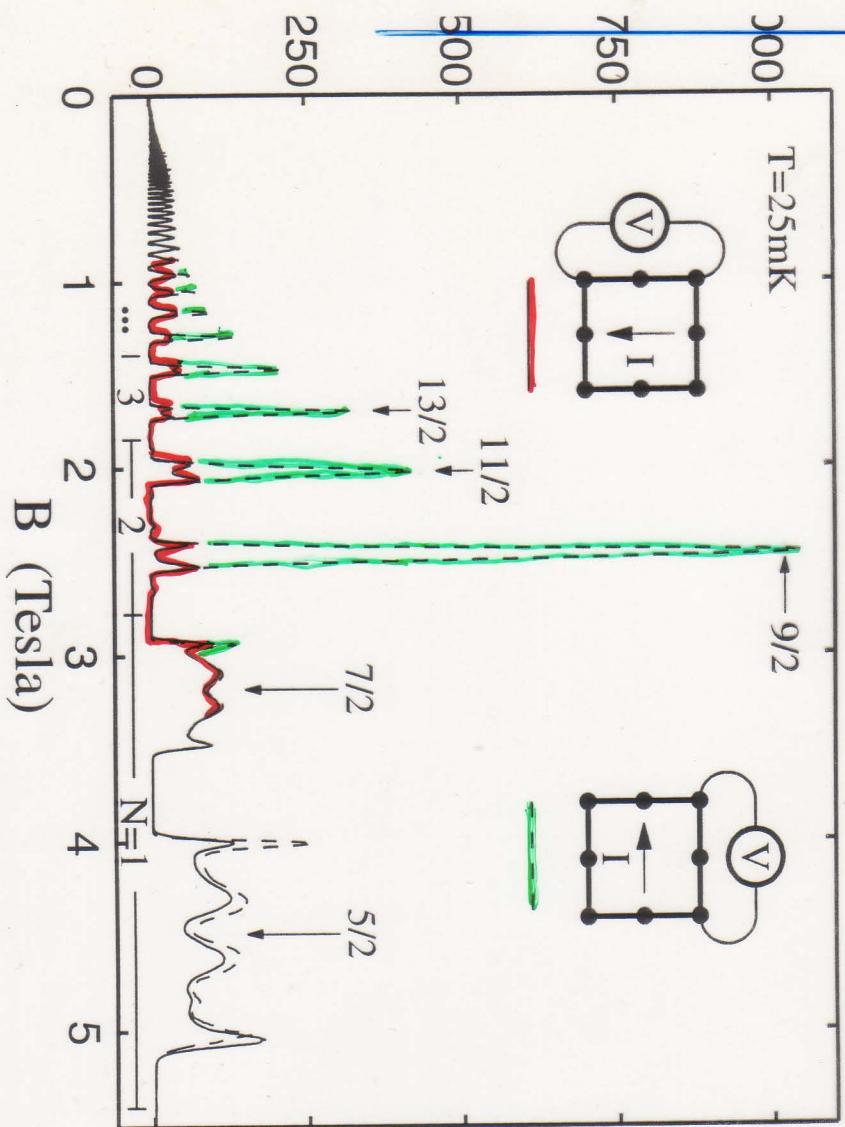
Recently there has been a great deal of interest in the properties of quasi-two-dimensional electron gases. !

Wigner Crystals, Stripes and Bubbles



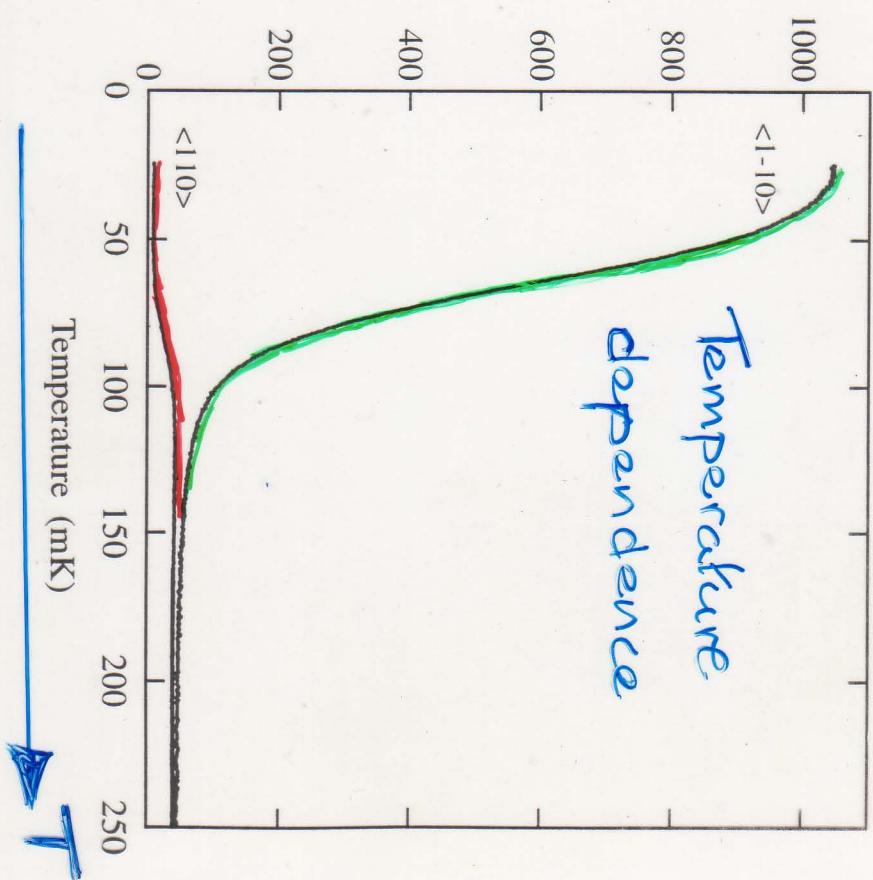
Anisotropic Conduction at Low Temperature in High Landau Levels

Longitudinal resistance



Lilly, Cooper, Eisenstein
Pfeiffer + West 1999

Longitudinal Resistances (Ω)

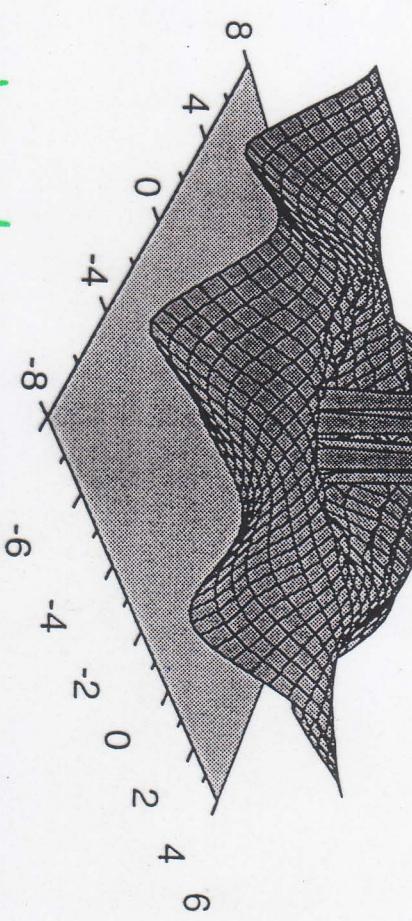


Density Correlations by Exact Diagonalisation

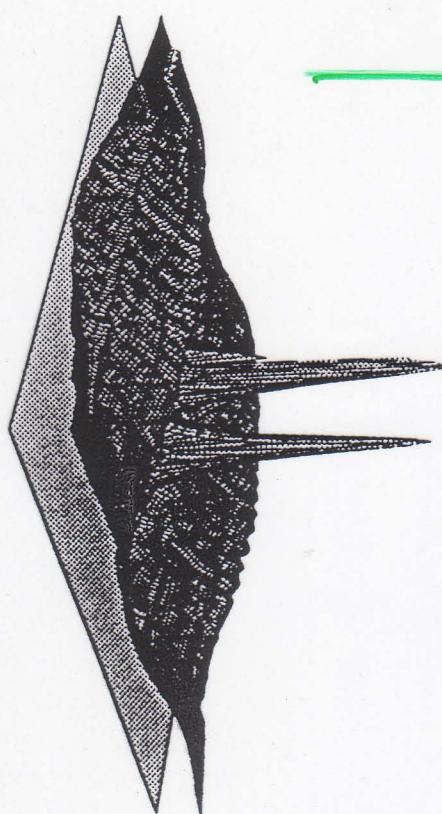
8 electrons on torus

Rezai, Haldane and Yang 2000

$n=2$ Landau level



$S(q)$
 $\frac{1}{4}$ filling



$S(q)$
 $\frac{1}{2}$ filling

Quantum Hall Ferromagnets

Cyclotron, Zeeman

& Exchange energies

Typically

$$g^* \mu_B B \sim \frac{1}{4D} \times \hbar \omega_c$$

Spin waves

$$\text{Landau gauge basis states} \\ \psi_k(x,y) \sim e^{iky} e^{-\frac{i}{2}(x+k)^2}$$

Polarised ground state

$$|\Psi_0\rangle = \prod_k c_{k\uparrow}^\dagger |0\rangle$$

Spin wave excitation

Exchange \Rightarrow spin polarisation
even for $g^* = 0$

$$|q\rangle = \sum_k e^{ikqx} c_{k-qy}^\dagger c_{k\uparrow} |0\rangle$$

cf

$$\Psi = \prod_{i < j} (z_i - z_j) e^{-\frac{1}{4} \sum_k |z_k|^2}$$

Real space
picture

(+)

 v_{drift}

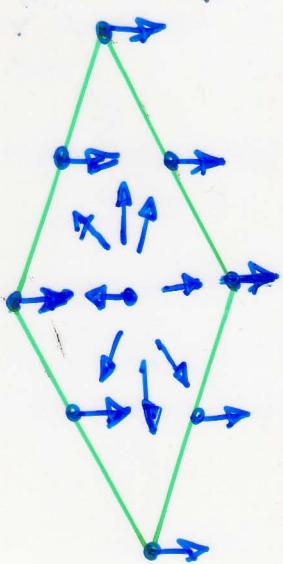
Topological Excitations

Berry phase modulates charge density

$$\delta\rho = \frac{e}{4\pi} \vec{S} \cdot (\partial_x \vec{S} \times \partial_y \vec{S})$$

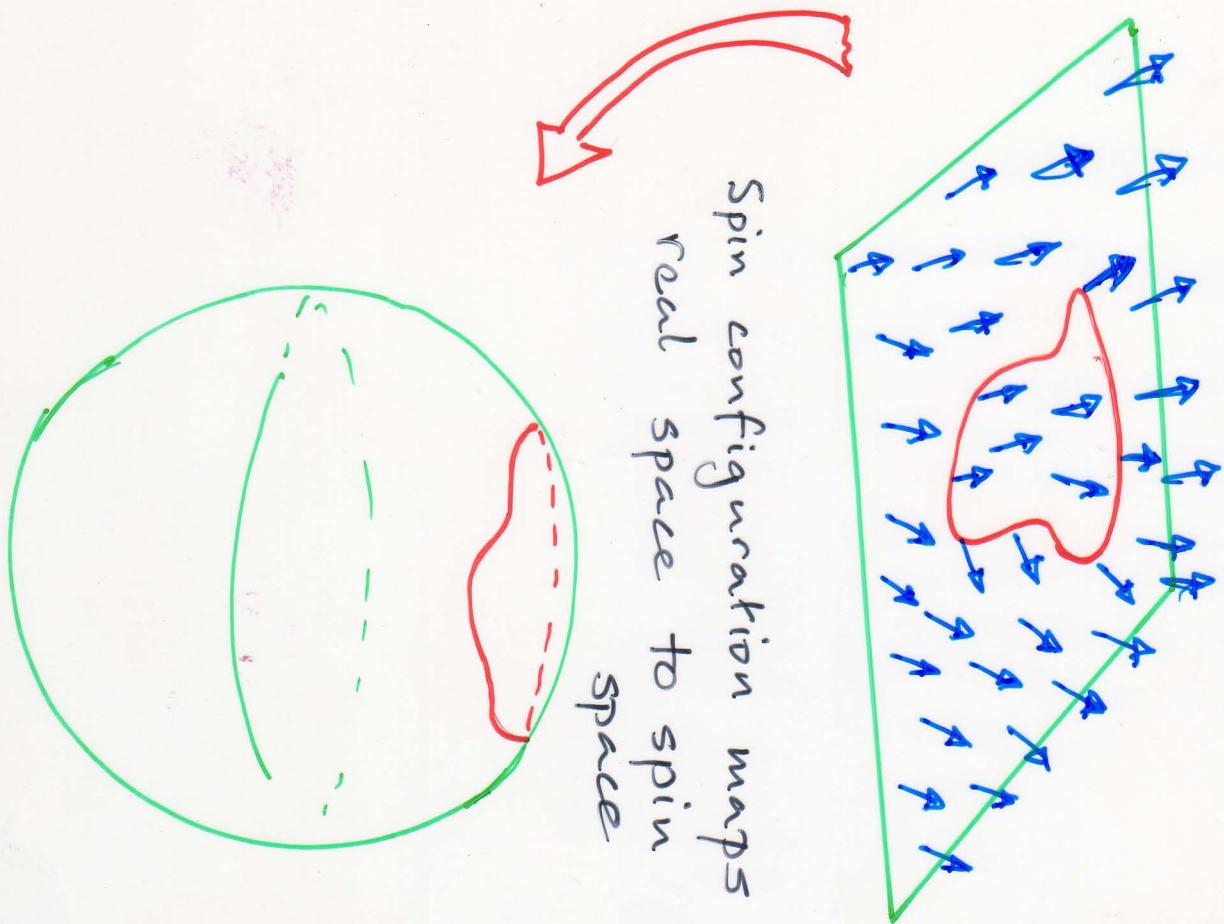
Skyrmion

$$\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \frac{1}{r^2 + a^2} \begin{pmatrix} 2a r \cos \phi \\ 2a r \sin \phi \\ r^2 - a^2 \end{pmatrix}$$



$$\delta\rho = -\frac{e}{\pi} \frac{a^2}{(\alpha^2 + r^2)^2}$$

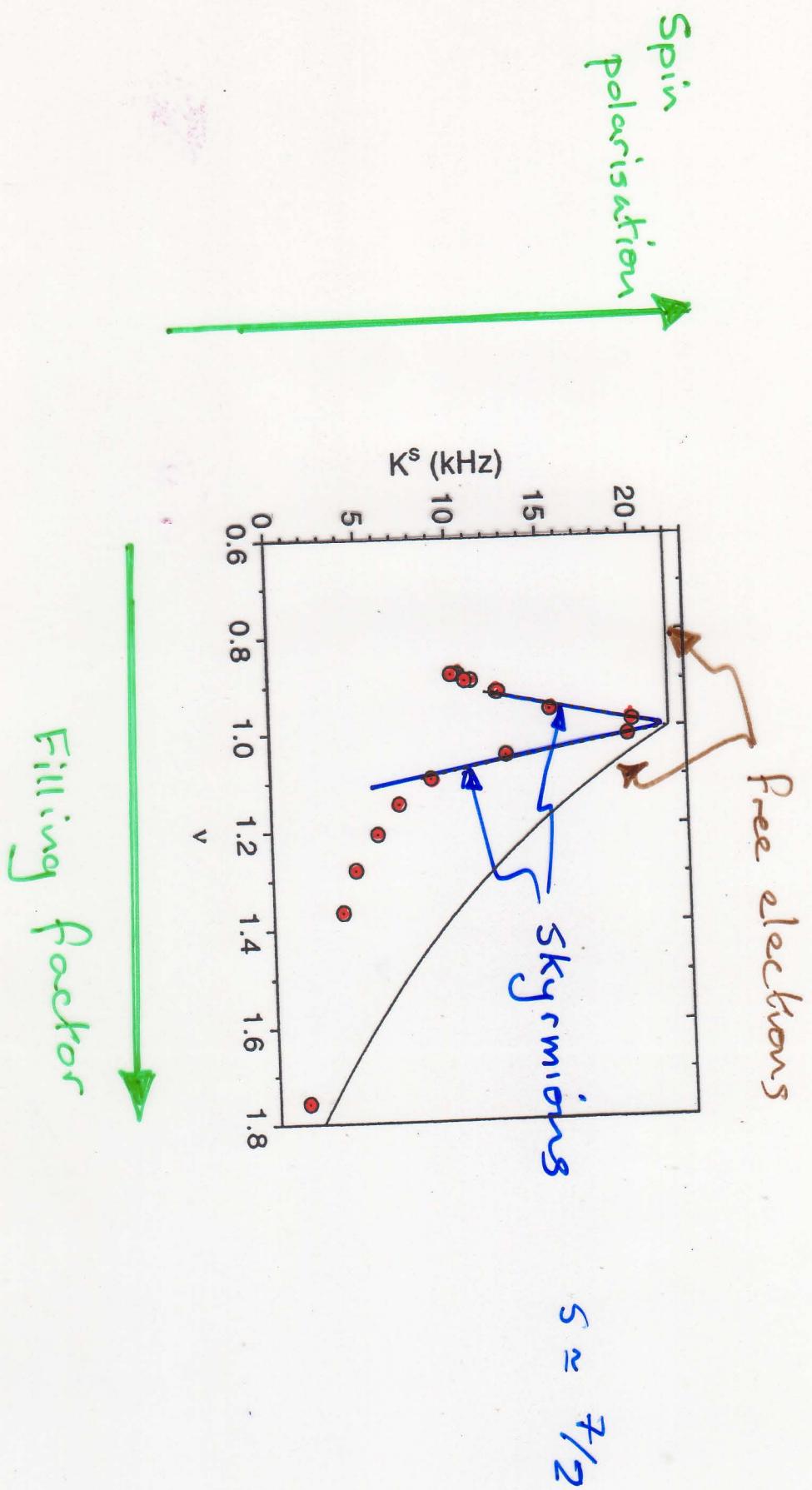
Spin configuration maps
real space to spin
space



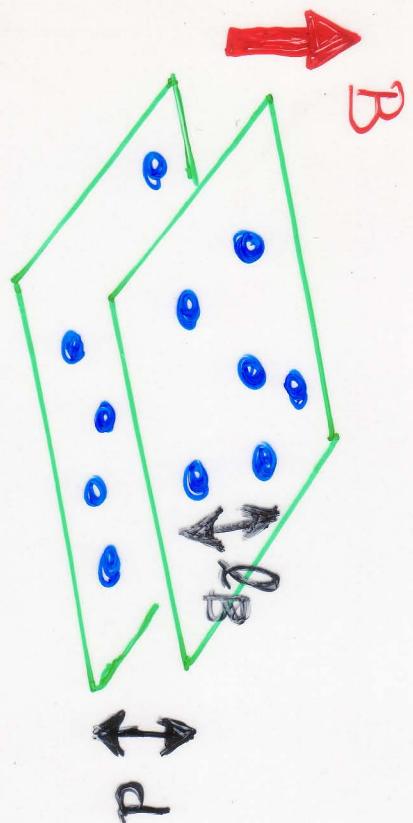
**Optically Pumped NMR Evidence for Finite-Size Skyrmions in GaAs Quantum Wells
near Landau Level Filling $\nu = 1$**

S. E. Barrett,* G. Dabbagh, L. N. Pfeiffer, K. W. West, and R. Tycko[†]
AT&T Bell Laboratories, 600 Mountain Avenue, Murray Hill, New Jersey 07974

(Received 19 December 1994)



Quantum Hall Bilayers at $\nu_{\text{Total}} \approx 1$

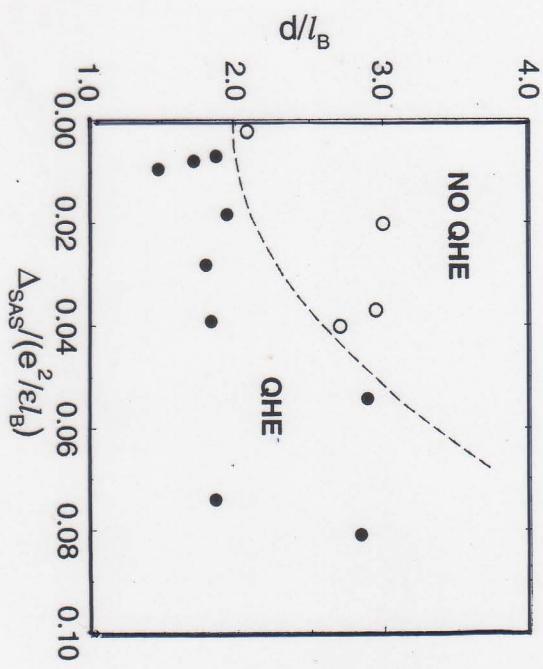


Parameters

$\frac{d}{l_B}$::
inter- vs intra-layer
correlations

$\frac{\Delta_{\text{SAS}}}{e^2/\epsilon l_B}$::
single particle
vs correlation
energies

Phase diagram



Descriptions of interlayer coherence

Easy plane ferromagnet

Pseudospins
 $\uparrow \sim$ layer # a
 $\downarrow \sim$ layer # b

Correlated state:

$$|\Psi\rangle = \prod_k (\cos \frac{\theta}{2} a_k^\dagger + \sin \frac{\theta}{2} e^{i\phi} b_k^\dagger) |0\rangle$$

Particle hole transformation
in one layer: $d_k^\dagger = a_k$

Condensate:

$$|\Psi\rangle = \prod_k (u + v d_k^\dagger b_k^\dagger) |0\rangle$$

Cost of smooth changes
in pseudo spin orientation

Charge balance

$$|u| = |v|$$

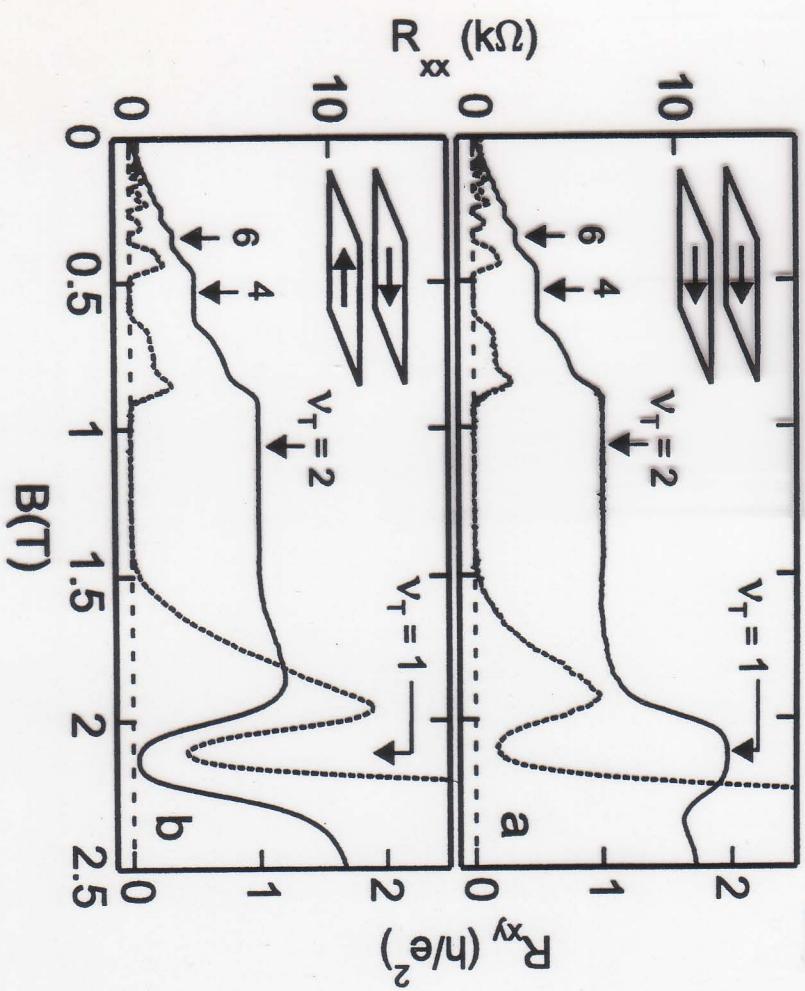
$$E \sim \int d\vec{r} \left\{ \frac{J}{2} |\nabla \vec{S}|^2 + D(S^z)^2 - t S^x \right\}$$

$|\vec{S}|=1$
 exchange tunneling
 charging

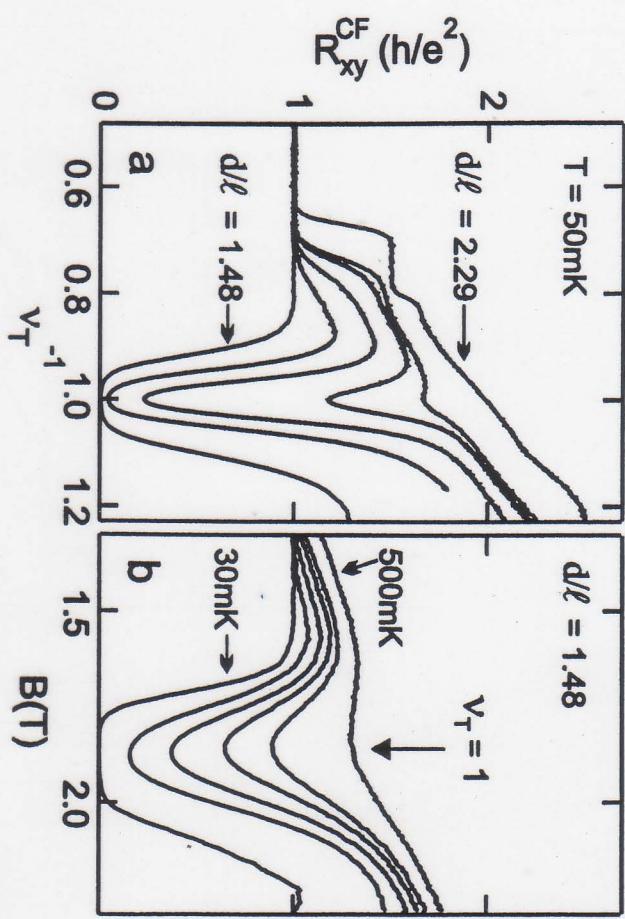
Observation of excitonic Superflow

Kellogg, Eisenstein et al
PRL (2004)

Counter flow vs Parallel flow



Dependence on T & d/ℓ



All components of ~~flow~~
vanish for counter flow

Lecture II: Summary

Ordering in high Landau levels

Crystallisation vs Laughlin states

Exchange energy and spin ordering

Spin textures and charge density

Topological excitations

Bilayers

Pseudospin order/exciton condensation

Seminar: Interactions and Transport Between Coupled Quantum Hall Edge States

Joe Tomlinson (Oxford)

J-S Caux (Amsterdam)

John Chalker (Oxford)

Outline:

Edge states in multilayer quantum Hall systems

Overview of experiments

Theory for coupled edge states in multilayer systems

Past work on multilayer quantum Hall systems and chiral metal:

JTC + Dohmen, PRL (1995)

Balents and Fisher, PRL (1996)

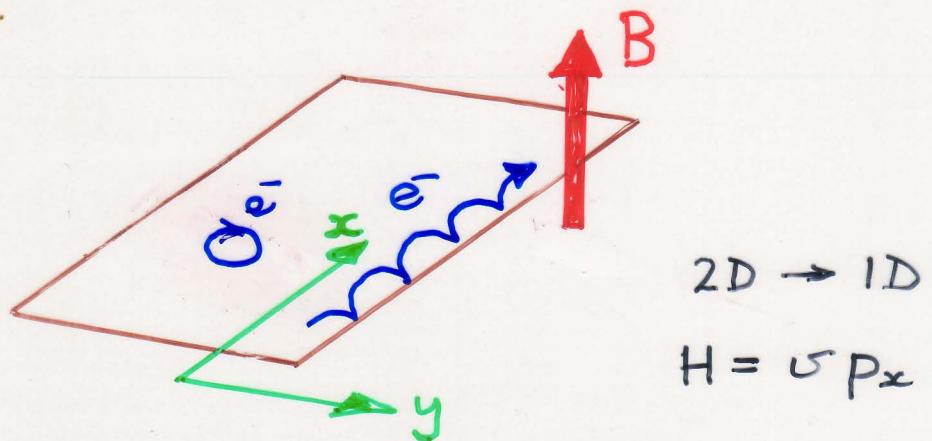
Cho, Balents and Fisher, PRB (1997)

Betouras + JTC, PRB (2000)

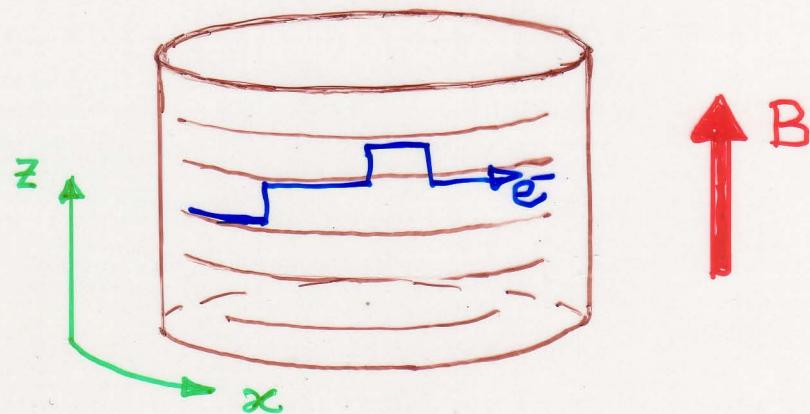
Experiments: Gwinn et al, UCSB

Edge States in Multilayer QH Systems

Single layer



Multilayer



$$H = H_{\text{edge}} + H_{\text{imp}} + H_{\text{int}} + H_{\text{hopping}}$$

$$H_{\text{edge}} + H_{\text{imp}} = \sum_n \int dx \psi_n^\dagger(x) [-iv\partial_x + V_n(x)] \psi_n(x)$$

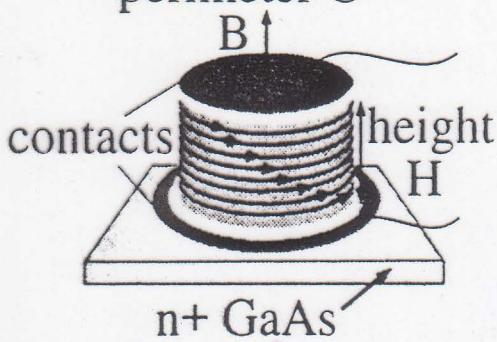
$$H_{\text{int}} = \sum_{n,m} \int dx \int dx' \rho_n(x) U_{n-m}(x - x') \rho_m(x')$$

$$\rho_n(x) = \psi^\dagger(x) \psi(x)$$

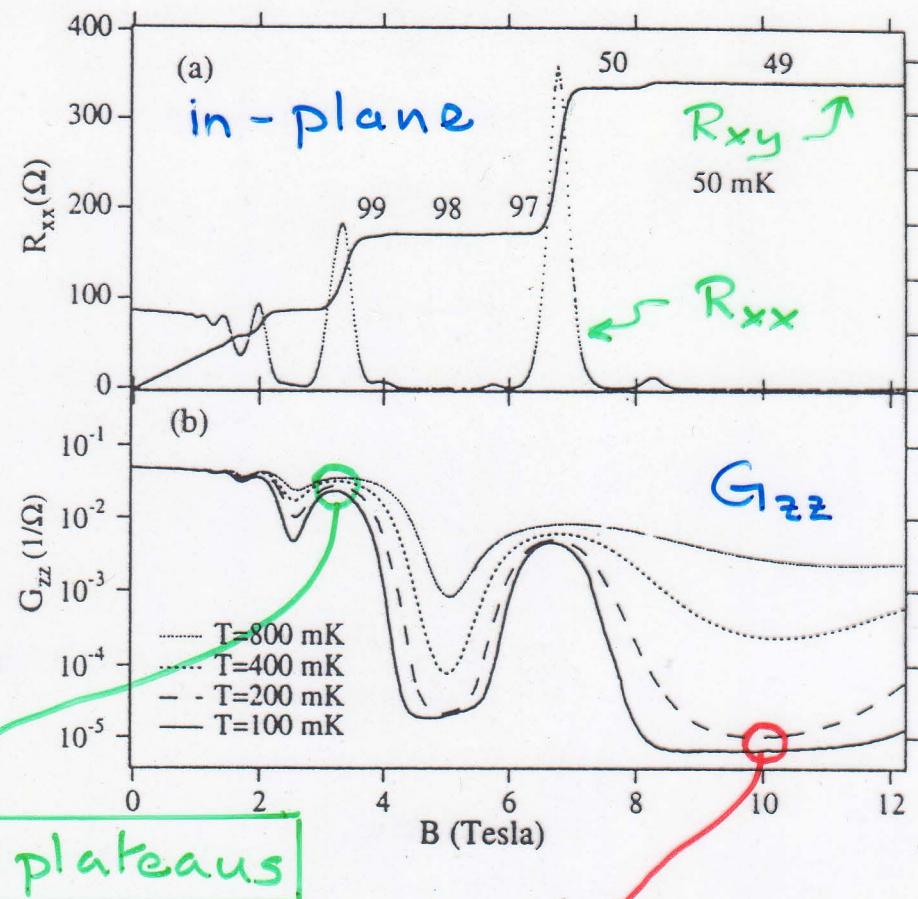
$$H_{\text{hopping}} = t_\perp \sum_n \int dx [\psi_{n+1}^\dagger(x) \psi_n(x) + \text{h.c.}]$$

Focussing on Surface States

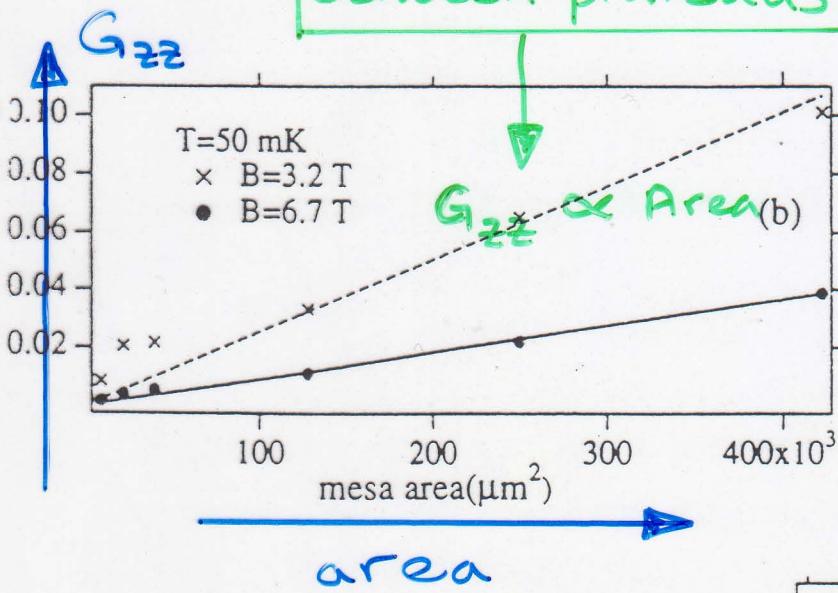
(c) Mesa, area A
perimeter C



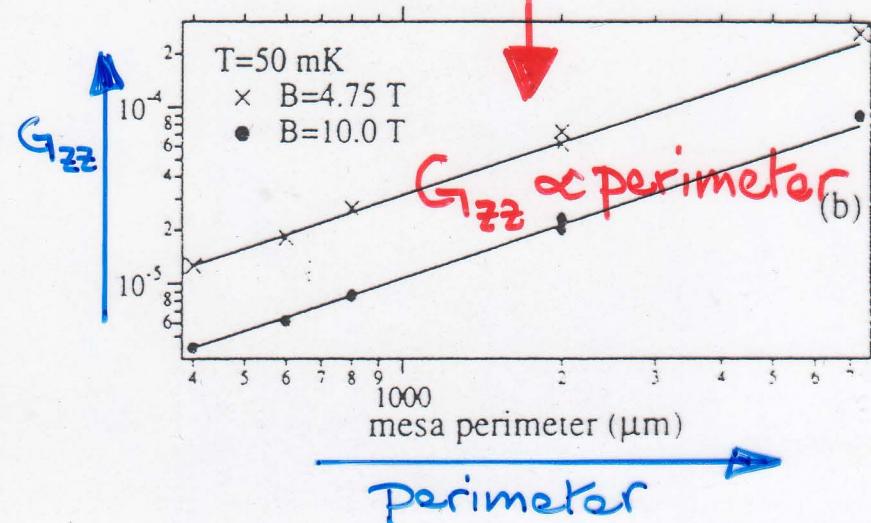
Vertical
transport: G_{zz}



between plateaus



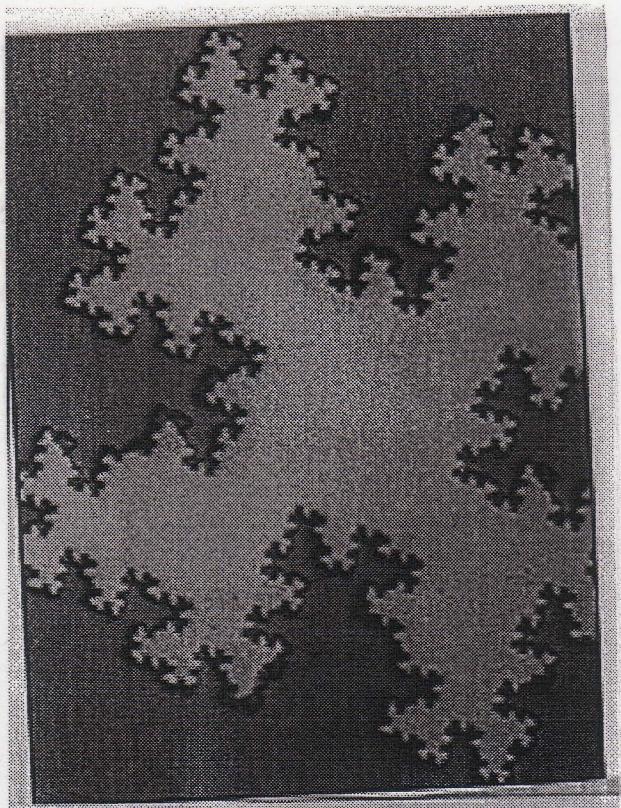
In plateaus



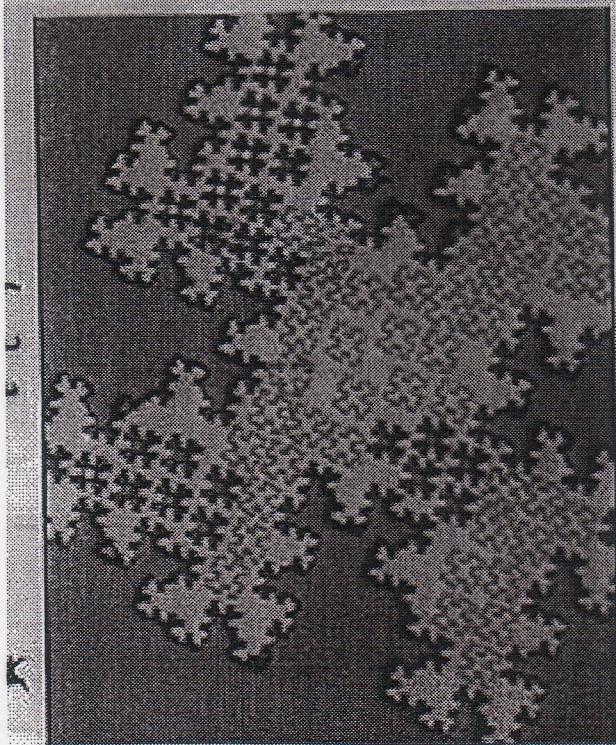
Druis et al 1998

UCSB

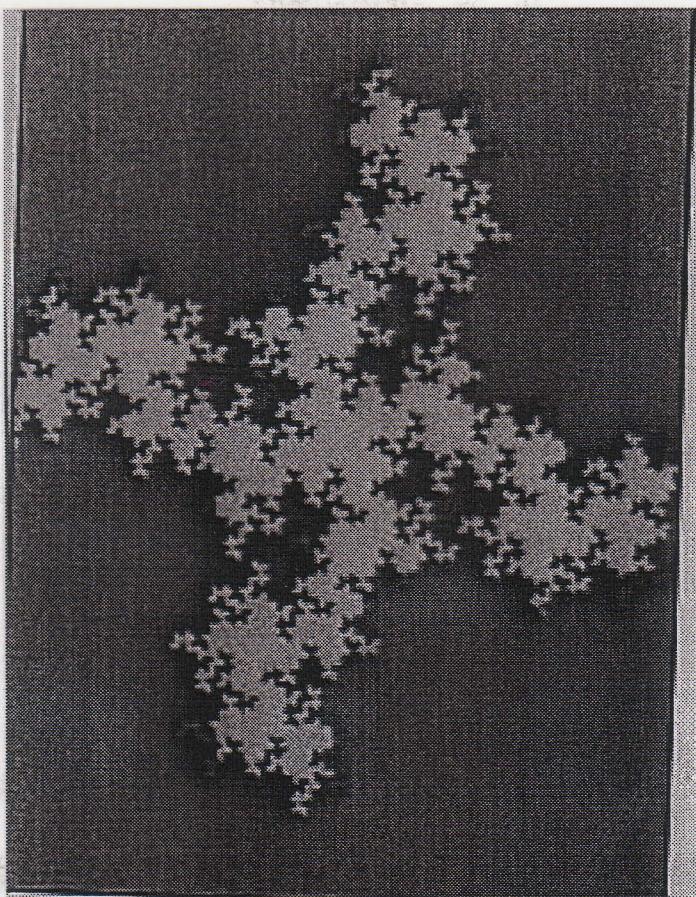
Samples for grinding surface states



Sample F1



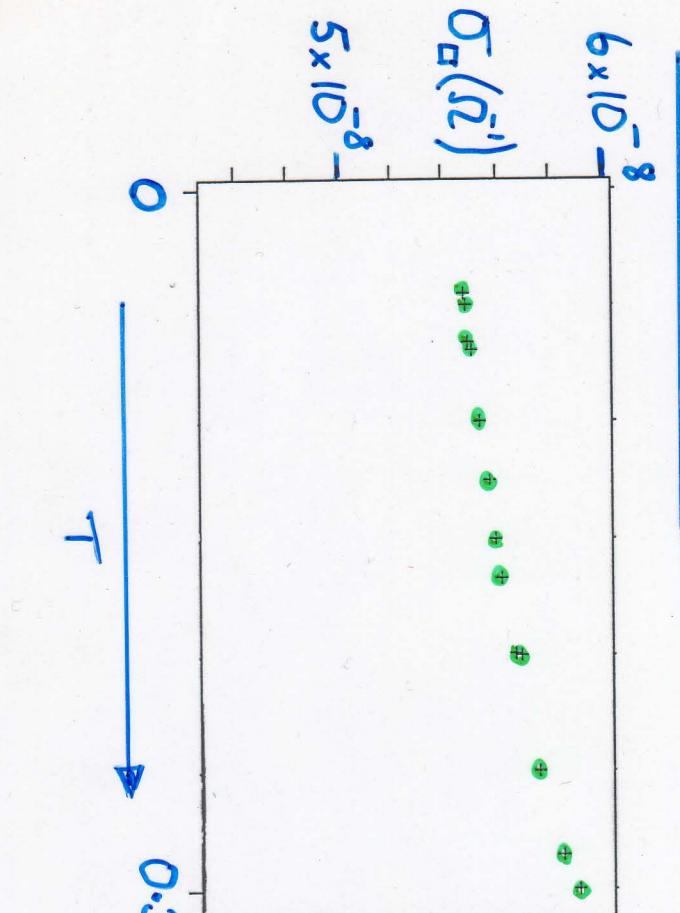
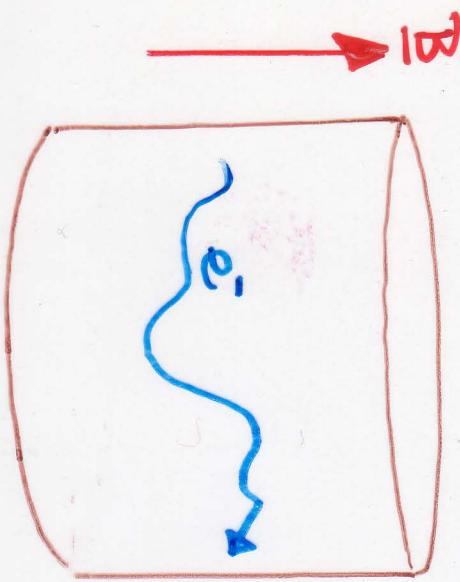
Sample F3



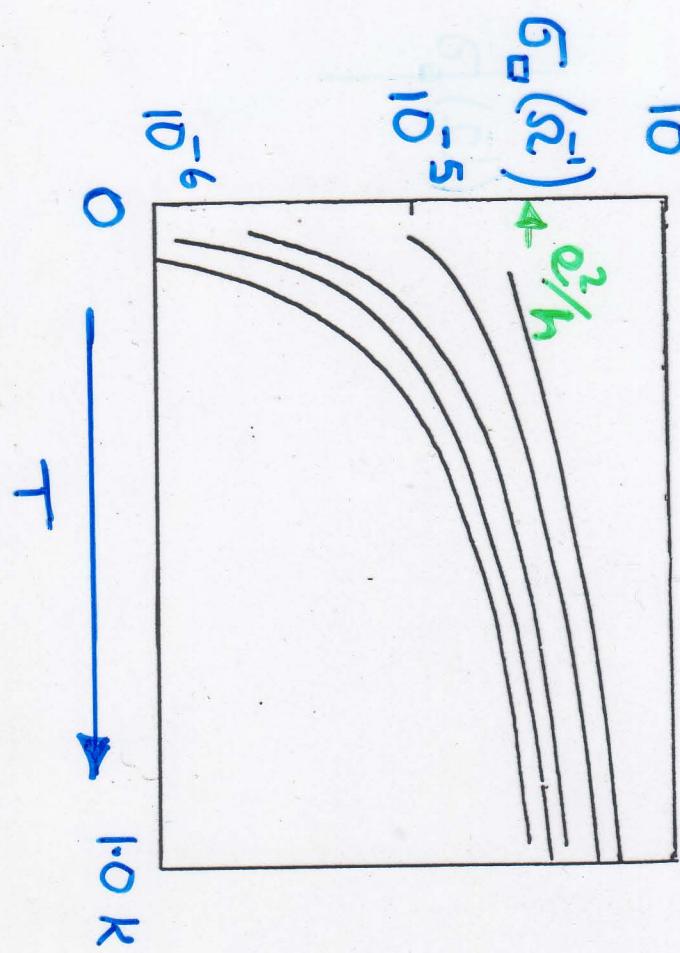
Sample F5

Special Features of Chiral Metal

Chiral Metal



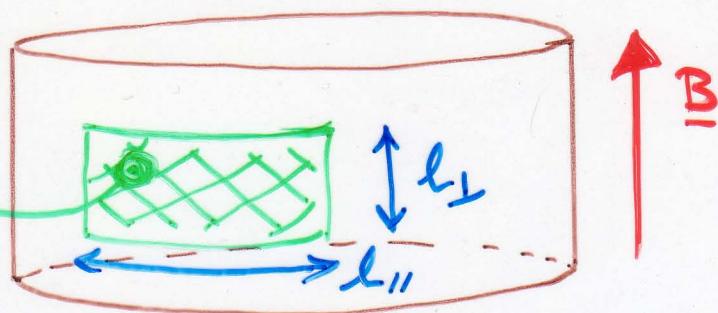
Conventional 2D MOSFET



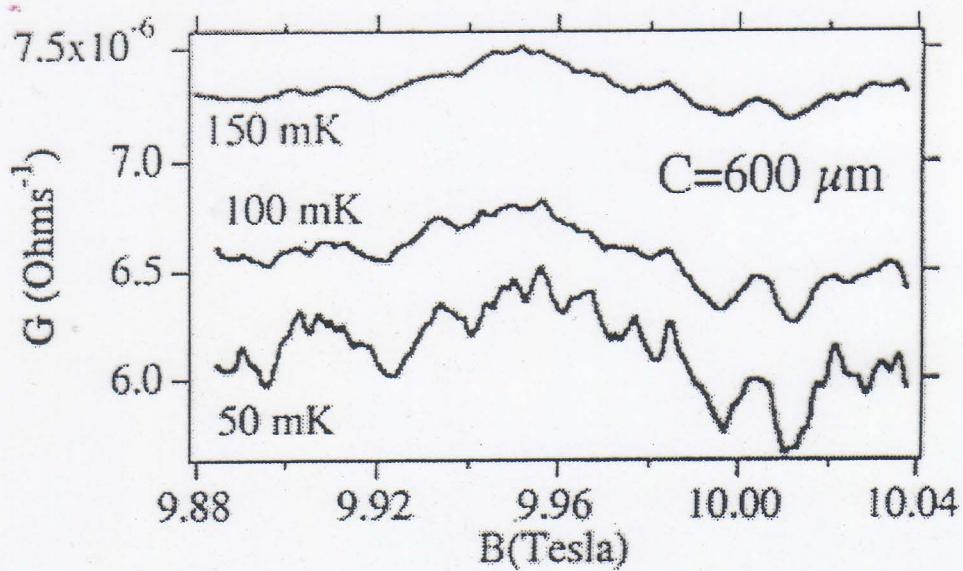
- No backscattering by disorder
 - no localisation

Conductance fluctuations as a Probe

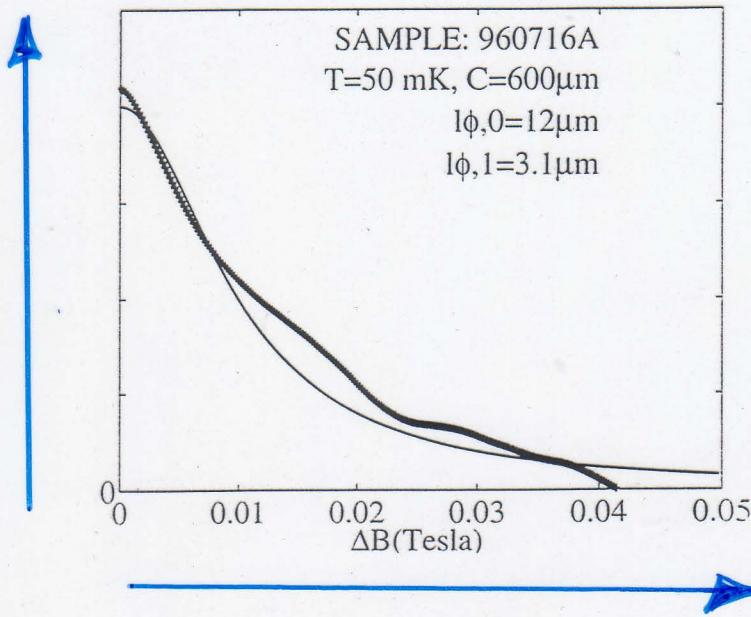
phase-coherent region



Mesoscopic fluctuations



$$\langle \delta G(B) \delta G(B + \Delta B) \rangle$$

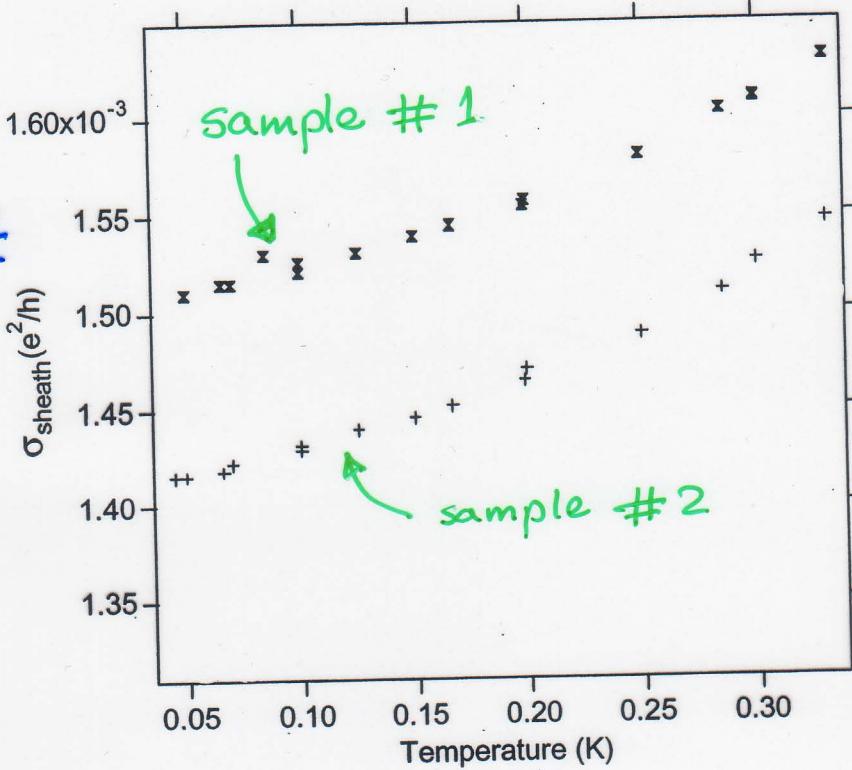


Inelastic scattering length
 $l_{\parallel} \approx 3 \mu\text{m}$

Recent Experiments

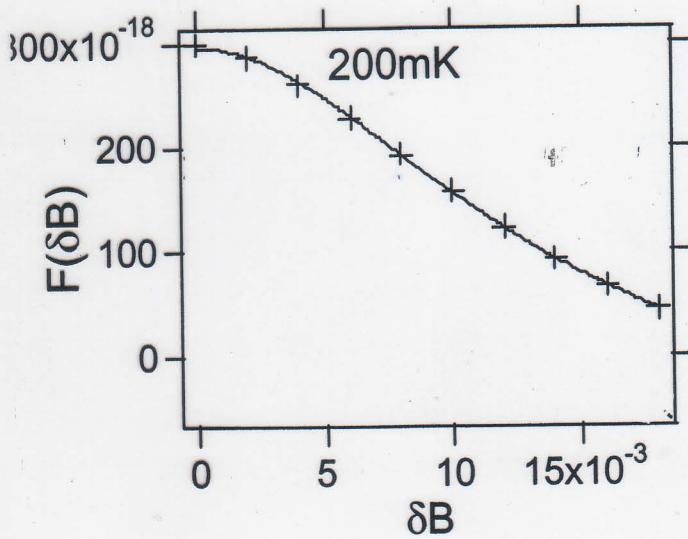
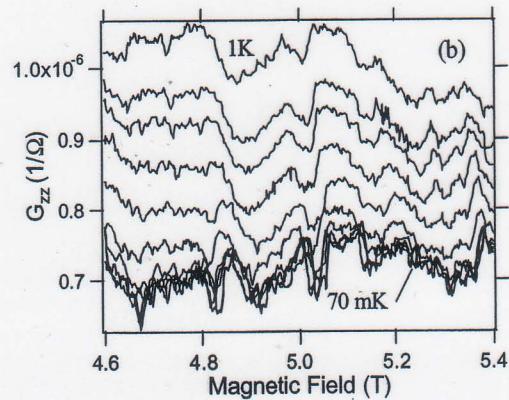
Walling et al
UCSB (2004)

Temperature dependence of conductance

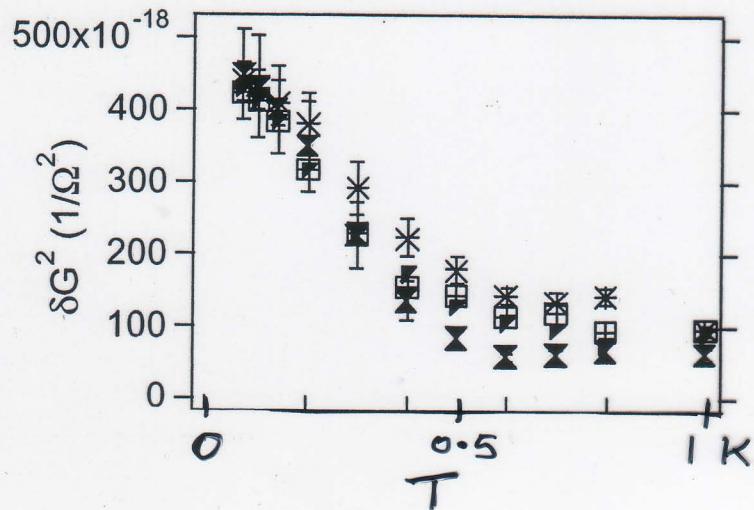


Conductance fluctuations

Correlator



T-dependence



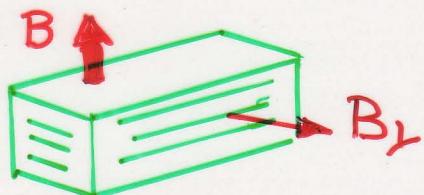
Scales in experimental system

Sample

Layer spacing $a = 30\text{nm}$
Number of layers: 50 - 100

Disorder

From transverse magnetoresistance
 $l_{\text{elastic}} \sim 40\text{nm}$



Interactions

From amplitude of conductance fluctuations
 $l_{\text{inelastic}} \sim 0.5 - 3\mu\text{m}$
depending on sample and temperature

Temperature

Thermal length

$$L_T = \hbar v / k_B T = 10\mu\text{m}$$

at 100 mK for sharp confining potential

Tunneling

Tunneling length $l_\perp \sim 40\mu\text{m}$

From conductivity

$$\sigma = ne^2 D \rightarrow \sigma = (e^2/h) \cdot (a/l_\perp)$$

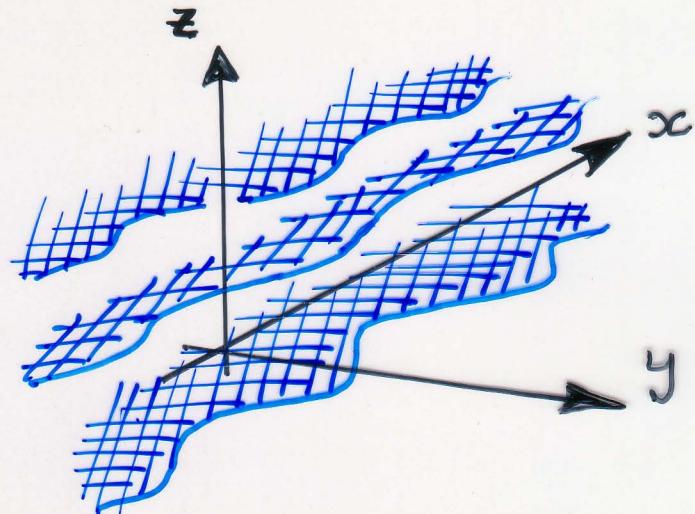
$$n = (ahv)^{-1} \quad \& \quad D = a^2/\tau_\perp \quad \& \quad l_\perp = v\tau_\perp$$

Classical treatment of surface modes without tunneling

Current

$$j_x(\vec{r}, t) = v\rho(\vec{r}, t)$$

$$j_z(\vec{r}, t) = 0$$



Continuity

$$\partial_t \rho(\vec{r}, t) = -\vec{\nabla} \cdot \vec{j}(\vec{r}, t) + \frac{\sigma_{xy}}{a} \mathcal{E}_x$$

electric field from $\rho(\vec{r}, t)$

Hall current to/from bulk

Surface magnetoplasmon dispersion

$$\omega(q_x, q_z) = vq_x \left[1 + \frac{\kappa}{(q_x^2 + q_z^2)^{1/2}} \right]$$

inverse screening length: $\kappa = \frac{ne^2}{2\epsilon\epsilon_0} = \frac{e^2}{2\epsilon\epsilon_0 ahv}$

experiment: $\kappa a \sim 2$ if confining potential sharp

Essentials of calculations

Disorder and gauge transformations

Single edge

$$H = \int dx \psi^\dagger(x) [-i\hbar v \partial_x + V(x)] \psi(x)$$

Remove $V(x)$ by $\psi \rightarrow e^{-i\theta(x)}\psi$

with $\theta(x) = (\hbar v)^{-1} \int^x V(x') dx'$

Phases appear in tunneling:

$$t_{\perp} \psi_{n+1}^{\dagger}(x) \psi_n(x) \rightarrow \underbrace{t_{\perp} e^{i[\theta_{n+1}(x) - \theta_n(x)]}}_{t_{\perp}(n, x)} \psi_{n+1}^{\dagger}(x) \psi_n(x)$$

Kubo formula for conductivity

- Get σ at leading order in t_\perp from $\langle \psi^\dagger \psi \psi^\dagger \psi \rangle$ calculated in system **without** tunneling

Bosonization

Get average conductance and mesoscopic fluctuations from

$$\langle \psi_n^\dagger(x, t) \psi_{n+1}(x, t) \psi_{n+1}^\dagger(0, 0) \psi_n(0, 0) \rangle$$

Only excitations are collective

- quantised surface magnetoplasmons

Transformation:

boson operator $b_{qx,n}^\dagger = i(\frac{2\pi}{Lq_x})^{1/2} \sum_{k_x} \psi_{k_x+q_x,n}^\dagger \psi_{k_x,n}$

fermion operator $\psi_n^\dagger(x) = e^{i\phi_n(x)}$

with $\phi_n(x) = -(\frac{2\pi}{L})^{1/2} \sum_q (e^{-iqx} b_{qn} + e^{iqx} b_{qn}^\dagger)$

Hamiltonian:

$$H_{\text{edge}} + H_{\text{int}} = \sum_{\vec{q}} \hbar \omega(q_x, q_z) b_{\vec{q}}^\dagger b_{\vec{q}}$$

Boson dispersion:

$$\omega(q_x, q_z) = v q_x [1 + \frac{\kappa}{(q_x^2 + q_z^2)^{1/2}}]$$

Results (1)

$$\langle \psi_n^+(x, t) \psi_{n+1}^-(x, t) \psi_{n+1}^+(0, 0) \psi_n^-(0, 0) \rangle = (2\pi)^2 e^S$$

with

$$S = -\frac{i}{\pi} \int_{-\pi}^{\pi} dq_z \int_0^\infty dq_x \frac{1 - \cos q_x}{q_x} \left\{ \coth \left(\frac{\beta \hbar \omega_q}{2} \right) [-\cos(q_x x - \omega_q t)] + i \sin(q_x x - \omega_q t) \right\}$$

Results

Dependence of $\sigma(T)$ on T

Non-interacting system:

$$\sigma = \frac{e^2}{h} \frac{2t_{\perp}^2 a l_{\text{elastic}}}{\hbar^2 v^2}$$

Interactions \rightarrow Dispersion

$$v \rightarrow v(q_x, q_z) \equiv \frac{\partial \omega(q_x, q_z)}{\partial q_x}$$

Coulomb Interactions:

$$\omega(q_x, q_z) = v q_x [1 + \frac{\kappa}{(q_x^2 + q_z^2)^{1/2}}]$$

For wide edge states

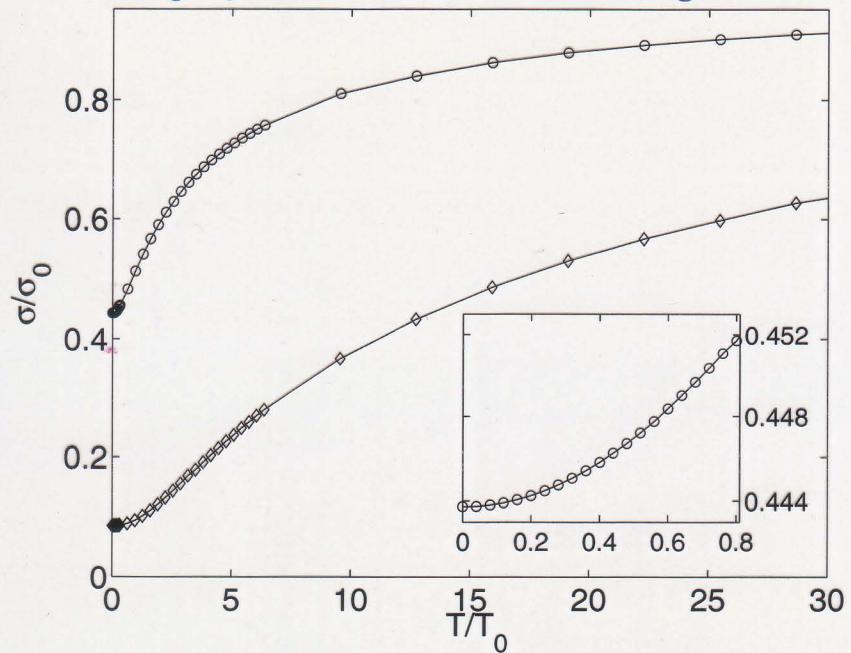
$$U_{m-n}(x - x') = \frac{e^2}{4\pi\epsilon\epsilon_0\sqrt{a^2(m-n)^2 + (x-x')^2 + w^2}}$$

and

$$\omega(q_x, q_z) = v q_x [1 + \frac{\kappa e^{-w|q|}}{(q_x^2 + q_z^2)^{1/2}}]$$

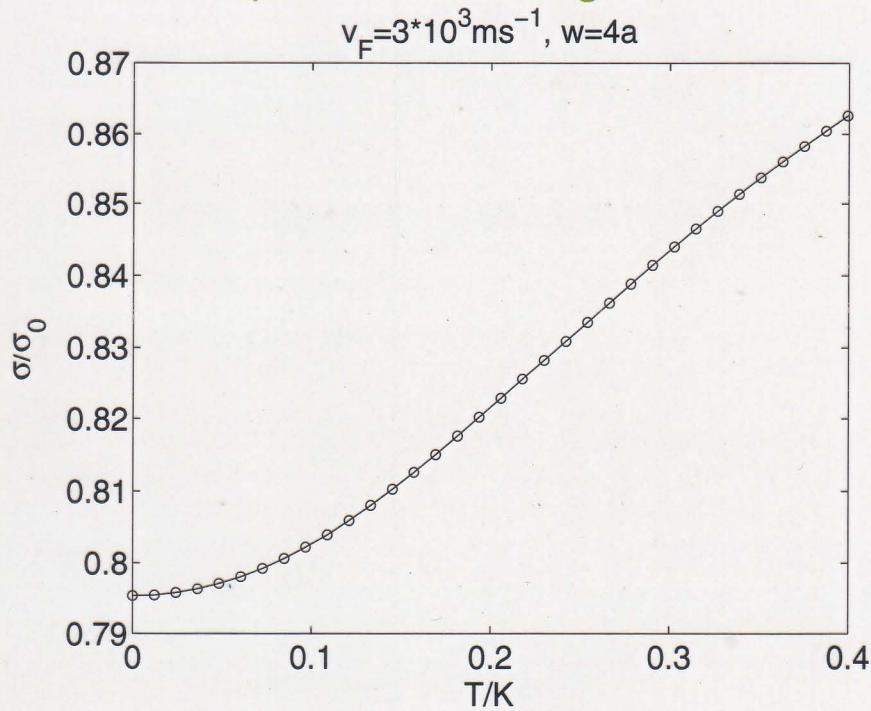
Dependence of $\sigma(T)$ on T

Steep edge potential \rightarrow Narrow edges



Soft confining potential \rightarrow Wide edges

Match to experiment with edge width 120 nm



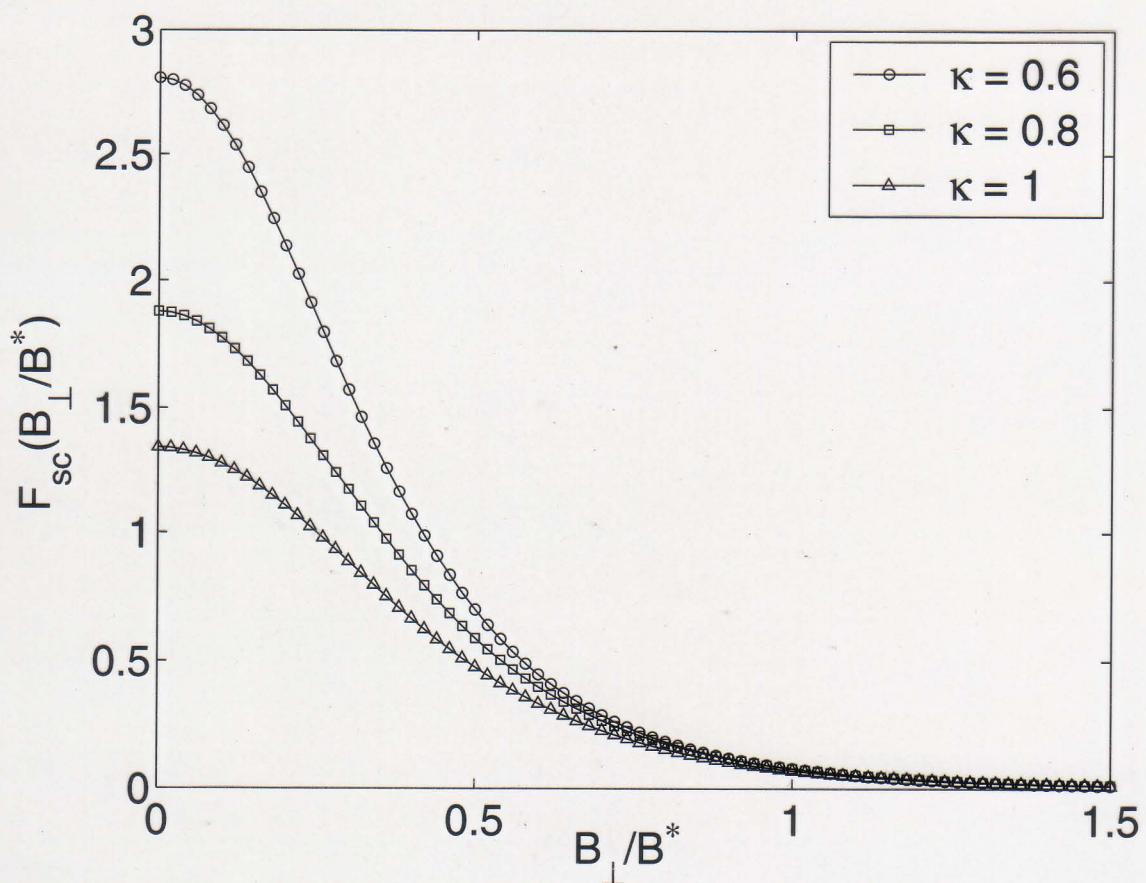
Conductance Fluctuations

At low T : scaling form

Function of ΔB and $L_T = \hbar v / k_B T$

$$\langle \delta g(B) \delta g(B + \Delta B) \rangle = \frac{g_0^2 L_T}{NL} F_\kappa(\Delta B / B^*)$$

$$B^* = \Phi_0 / a L_T$$



Summary

- Weakly coupled quantum Hall edges: can treat Coulomb interactions and disorder exactly using bosonization.
- Dependence of $\sigma(T)$ on T reflects full \vec{r} -dependence of Coulomb interactions.
- Conductance fluctuations suppressed with increasing T - despite coherence of bosonic excitations.