Eigenfunction correlation, random matrix theory and superconductivity near the Anderson transition

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Two-eigenfunction correlation function

$$C(E-E') = \frac{\sum_{n,m} V \int d^d r \left\langle |\Psi_n(r)|^2 |\Psi_m(r)|^2 \delta(E_n - E)\delta(E_m - E') \right\rangle}{\sum_{n,m} \left\langle \delta(E_n - E)\delta(E_m - E') \right\rangle}$$

Eigenfunction overlap at an energy separation E-E'

Why to bother?

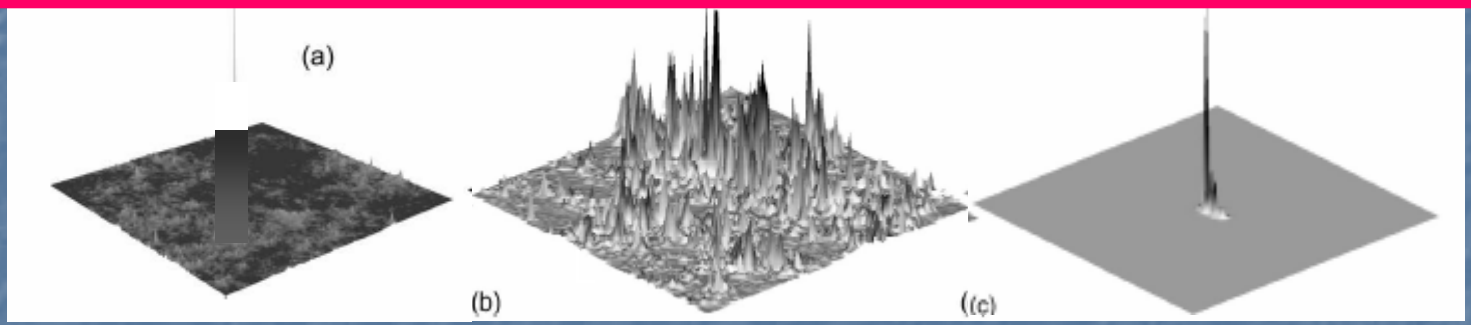
Matrix elements of local interactions,

e.g. local attraction in superconductivity

$$J_{ij} = g \sum_{r,|\varepsilon_{n,m}|<\omega_0} \Psi_n^2(r)\Psi_m^2(r)$$

Modification of states with increasing disorder

disorder

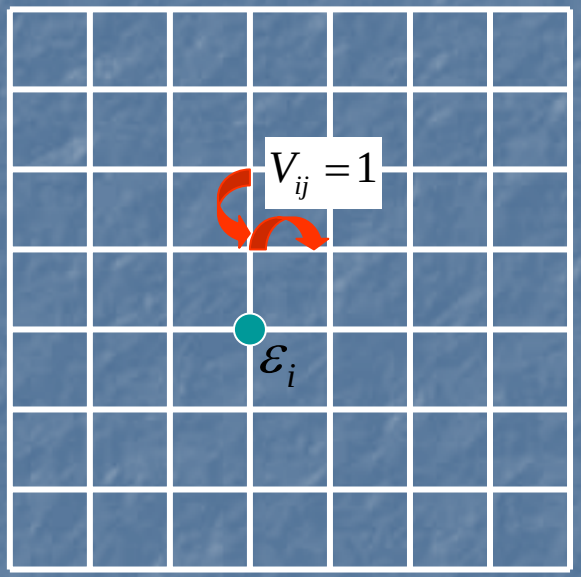


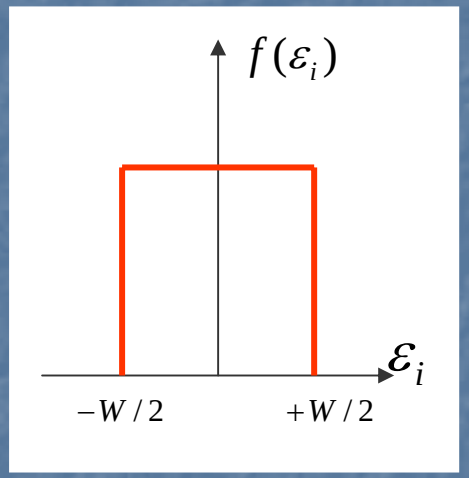
Extended states Critical states Localized states

How do the matrix elements change ?

The (standard) Anderson model

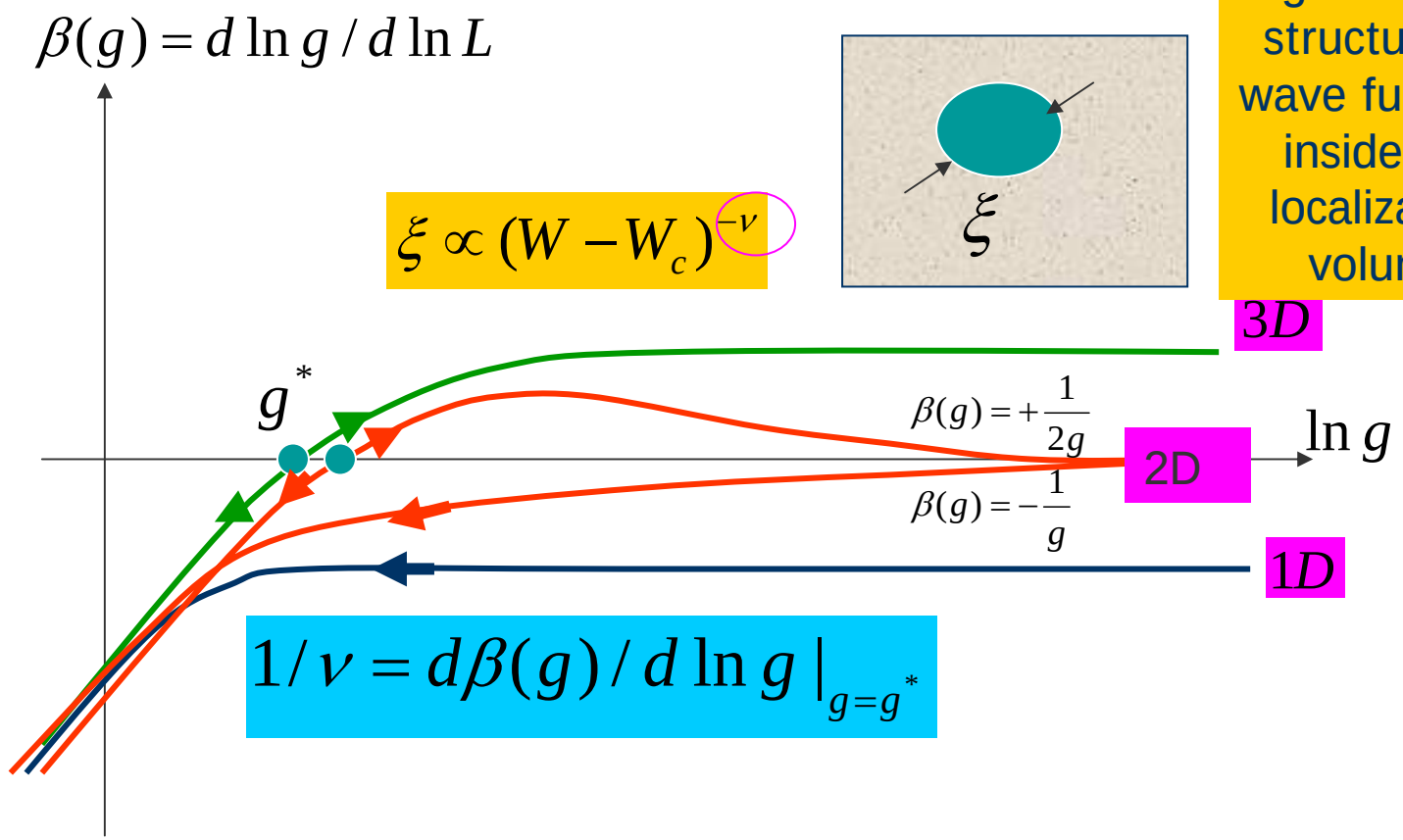
$$H = \sum_i \varepsilon_i c_i^+ c_i + \sum_{ij} V_{ij} c_i^+ c_j + c.c.$$





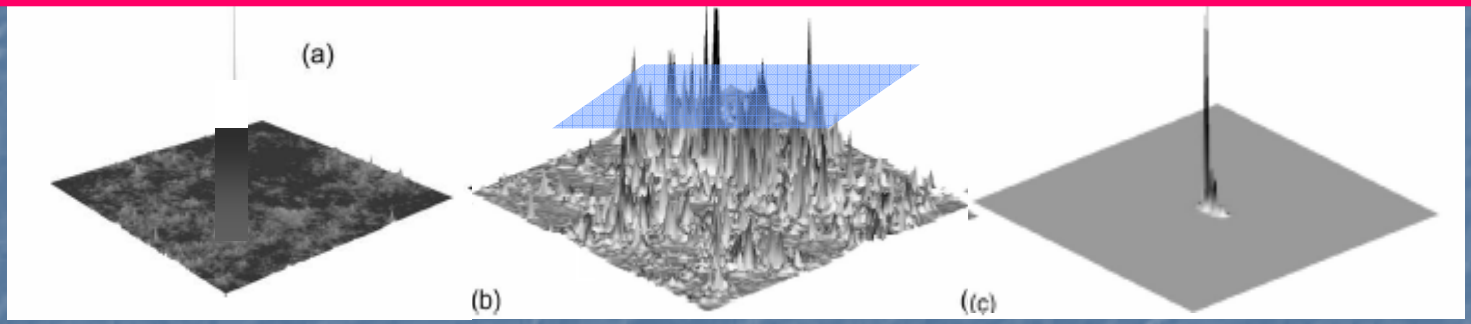
Shows localization for sufficiently strong disorder but difficult to treat both analytically and numerically

Scaling theory of localization: what it is about and what it ignores?



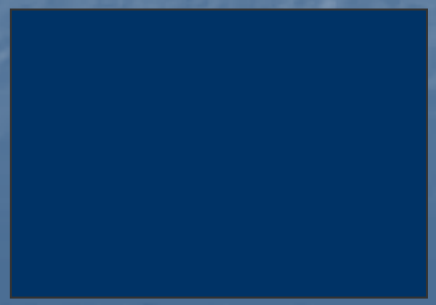
Extended, localized and critical eigenstates

disorder



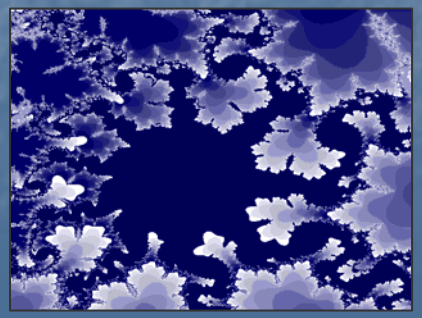
Extended states

$$\sum_r |\Psi_i(r)|^4 = \frac{1}{L^d} = \frac{1}{N}$$



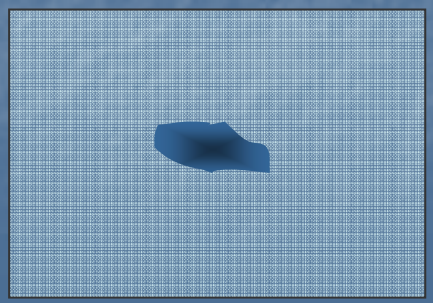
Critical multifractal states

$$\sum_r |\Psi_i(r)|^4 = \frac{1}{L^{d_2}} = \frac{1}{N^{d_2/d}}$$



Localized states

$$\sum_r |\Psi_i(r)|^4 = \frac{1}{\xi^d}$$



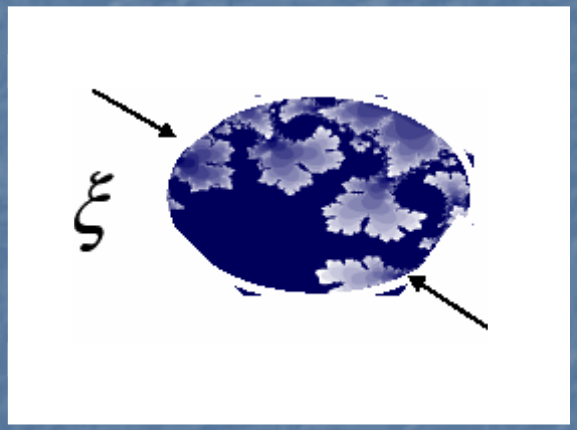
Why multi-fractal?

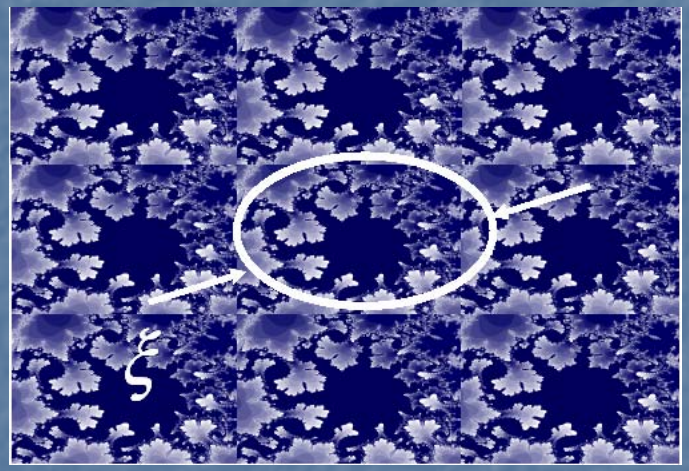
$$\sum_r |\Psi_i(r)|^{2q} = \frac{1}{L^{d_q(q-1)}}$$

$$d_q = d - \alpha q$$

Multifractal metal and insulator

Localization/correlation length ξ is much larger than the minimal length scale for fractality l_0





Multifractal insulator

Multifractal metal

Fractal texture persists in the metal and insulator

How do we know that?

Critical features in the eigenfunction correlation

Ideal metal and insulator

$$V \int d^d r \left\langle |\Psi_n(r)|^2 |\Psi_m(r)|^2 \right\rangle$$

Metal:

$$V \quad V \quad \frac{1}{V} \quad \frac{1}{V} = 1$$

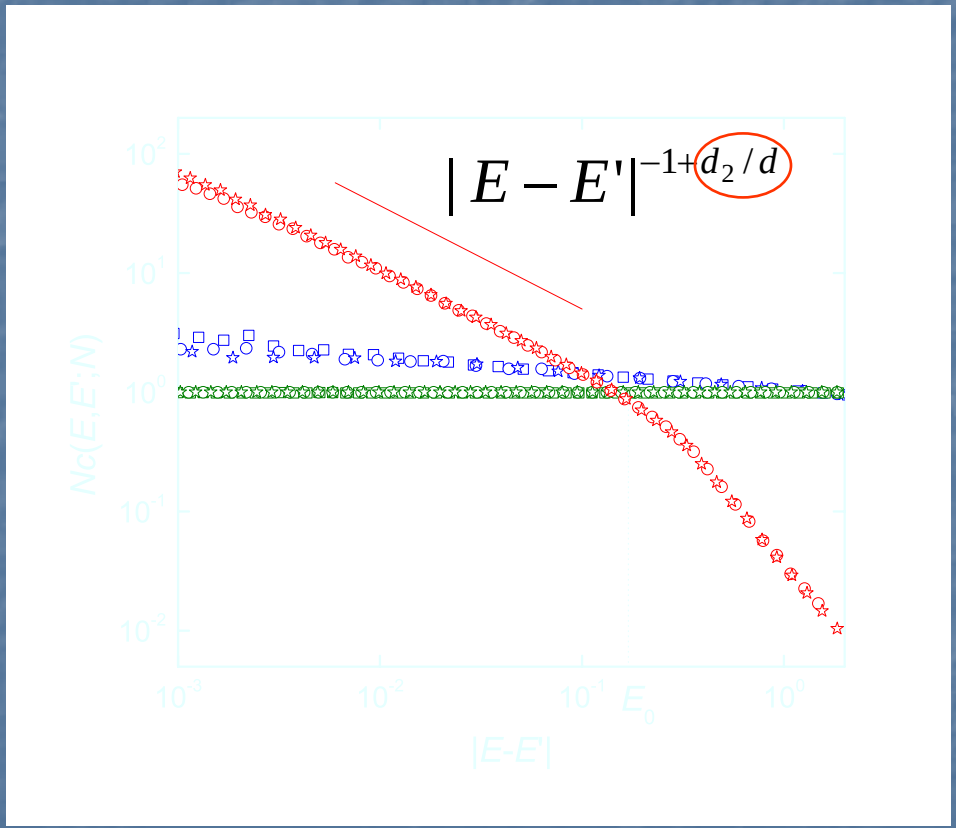
Small amplitude
100% overlap

Insulator:

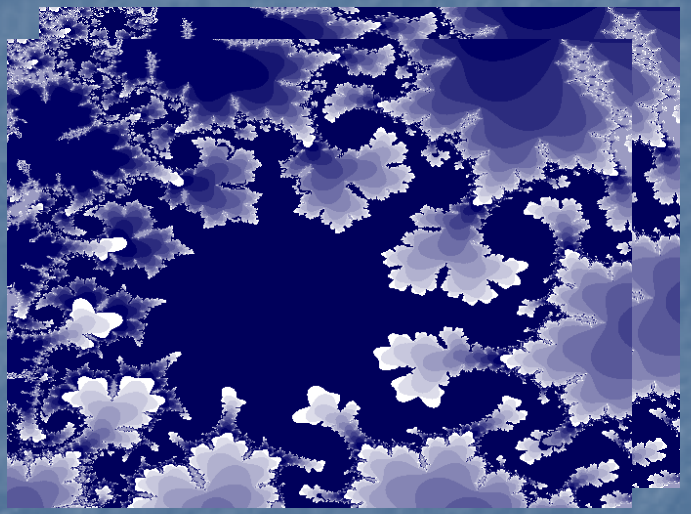
$$V \quad \xi^d \quad \frac{1}{\xi^d} \quad \frac{1}{\xi^d} \times \left(\frac{\xi^d}{V} \right) = 1$$

Large amplitude
rare overlap

Critical enhancement of correlations



Amplitude higher than in a metal but almost full overlap



States far away in energy are strongly correlated

Chalker's scaling: $|E - E'| < E_0$

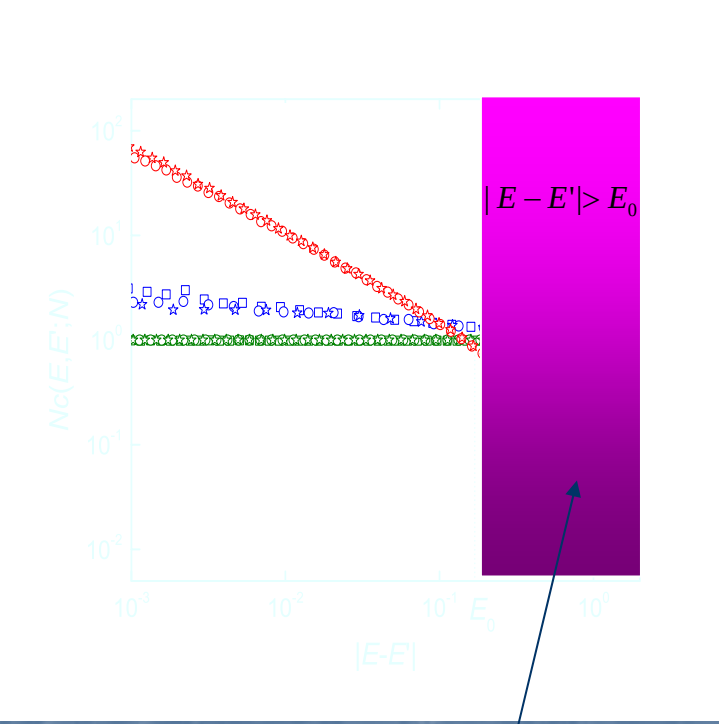
$$NC(E - E') = \left(\frac{E_0}{|E - E'|} \right)^{\mu}$$

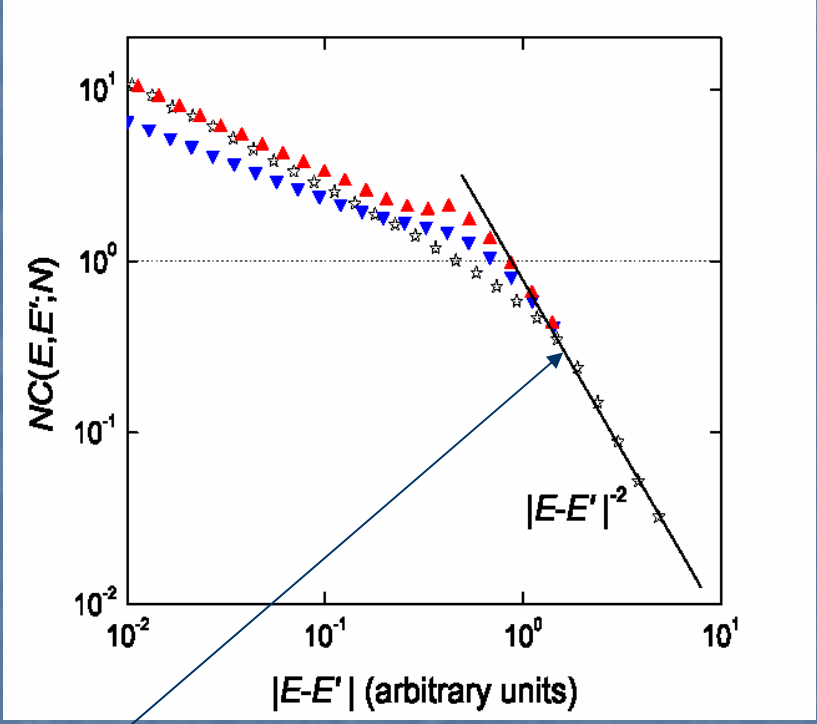
$$\mu = 1 - d_2 / d$$

$$0 < \mu < 1$$

Mismatch in the fractal structure grows slowly with the distance in the energy space

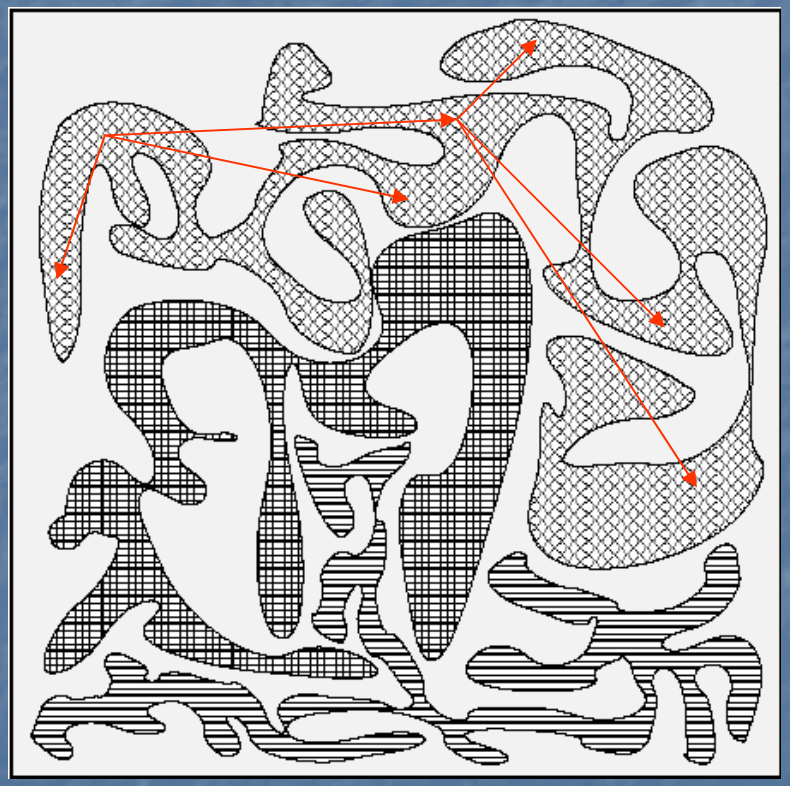
Self-avoiding of eigenfunctions at $E > E_0$





Overlap is smaller than for uncorrelated eigenfunctions

Stratification of space



Each shell consists of resonance sites for which $|E-E'|<V$

For $W = (\delta E_n) > V$ there are more than one shell which avoid each other in space

Intra-shell states overlap almost like in metal: enhancement of $C(\omega)$ at $\omega <$ bandwidth $= E_0$

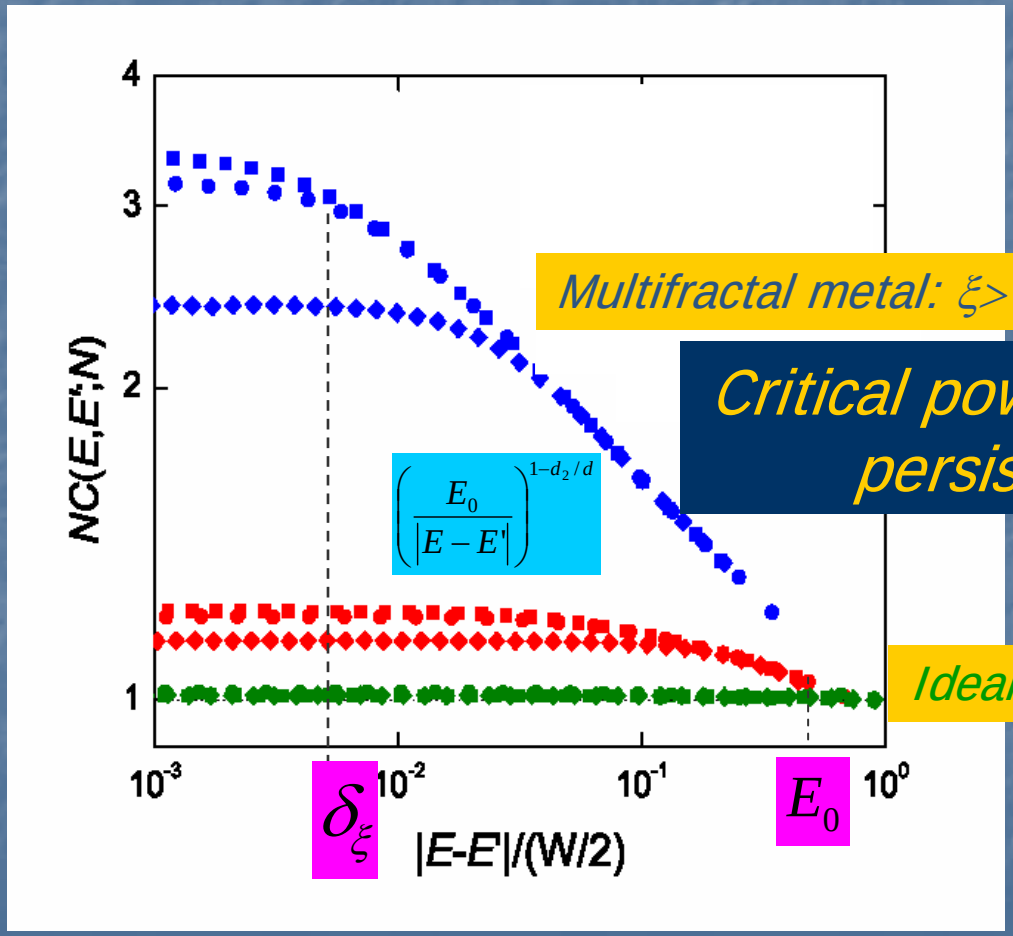
Inter-shell states avoid each other: $C(\omega)$ rapidly decreases for $\omega > E_0$.

From critical to off-critical states

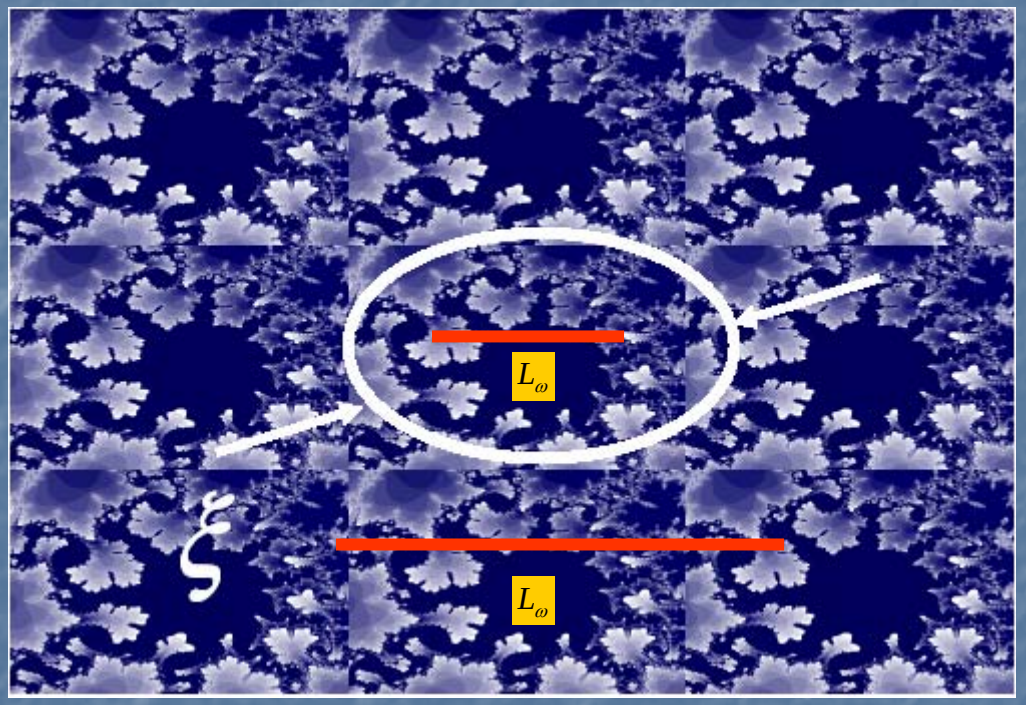
Two-eigenfunction correlation in 3D Anderson model (metal)

New length scale ℓ_0, new energy scale $E_0 = 1/\rho\,\ell_0^3$

$$\xi = \left| \frac{W_c}{W_c - W} \right|^{\nu}$$



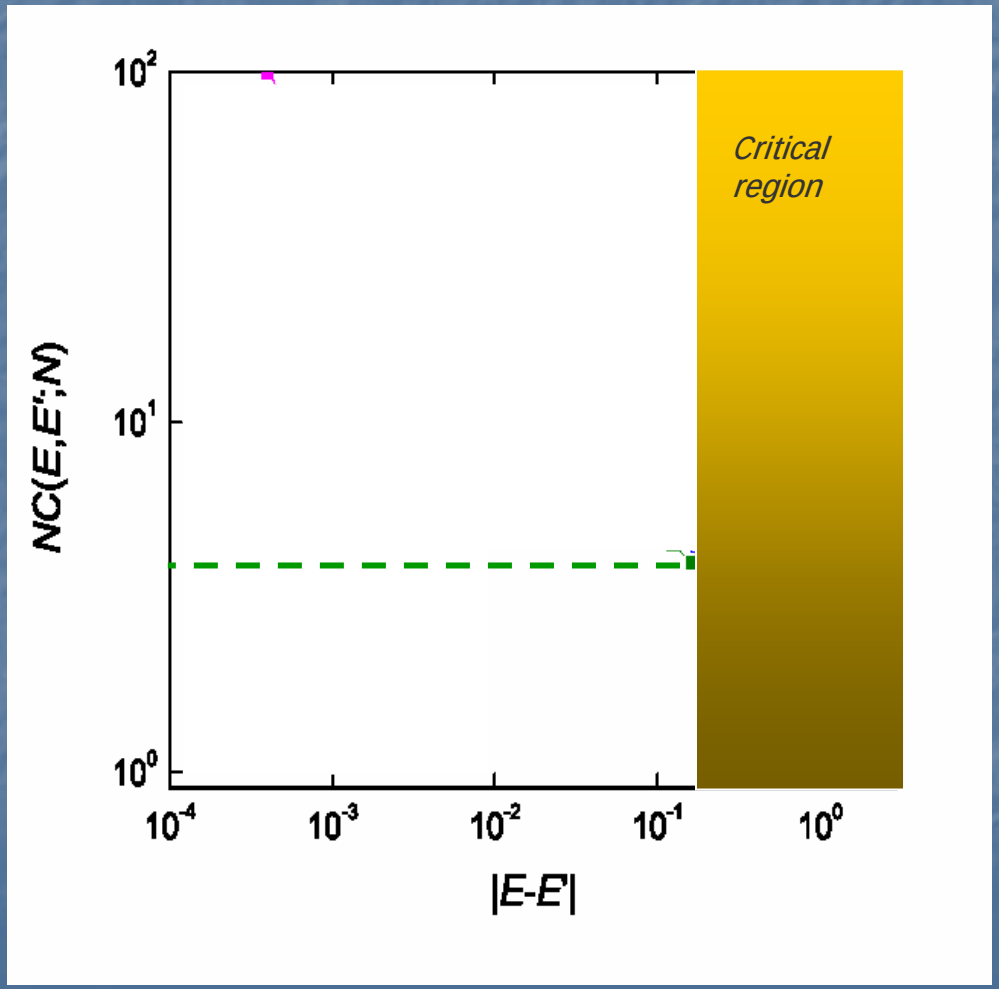
Dynamical length scale



$$L_\omega = \left(\frac{1}{\rho\omega} \right)^{\frac{1}{d}}$$

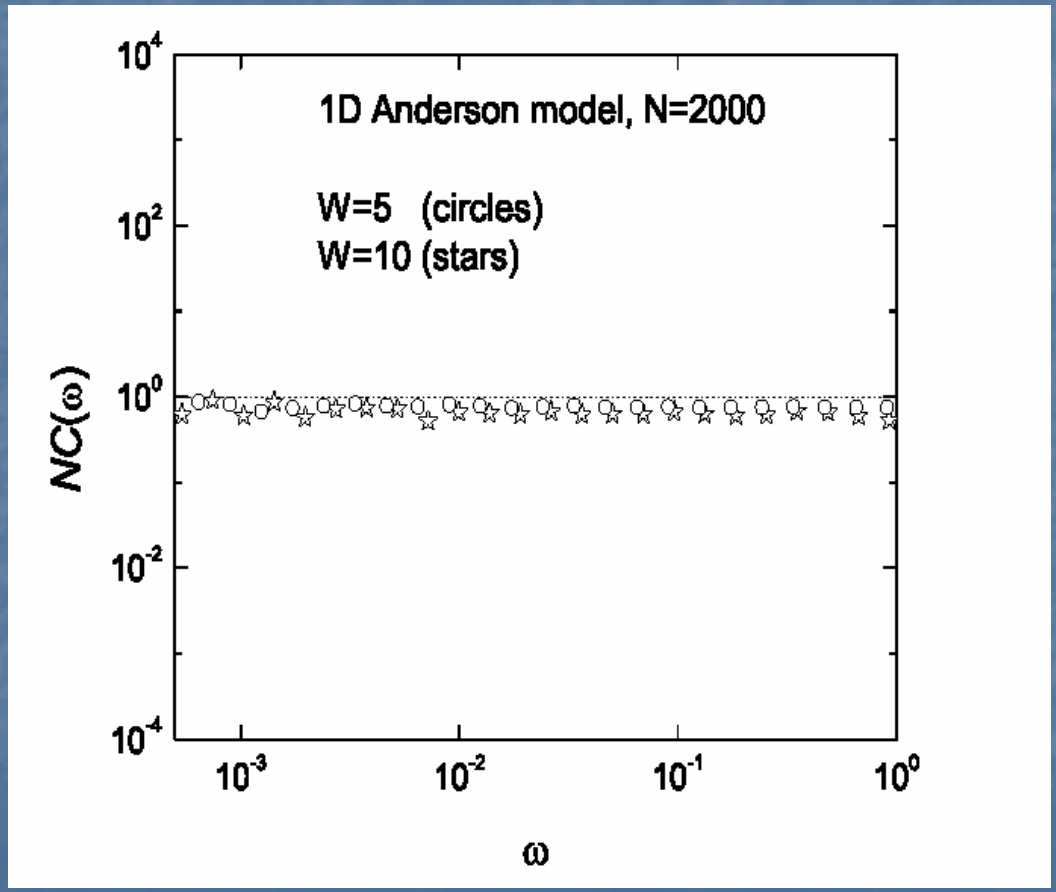
Dynamical length

Two-eigenfunction correlation in 3D Anderson model (insulator)



No ideal insulator even for very strong disorder!

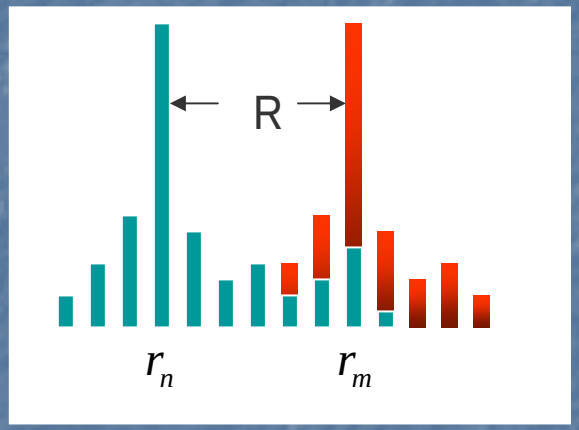
Two-eigenfunction correlation in 1D Anderson model (insulator)

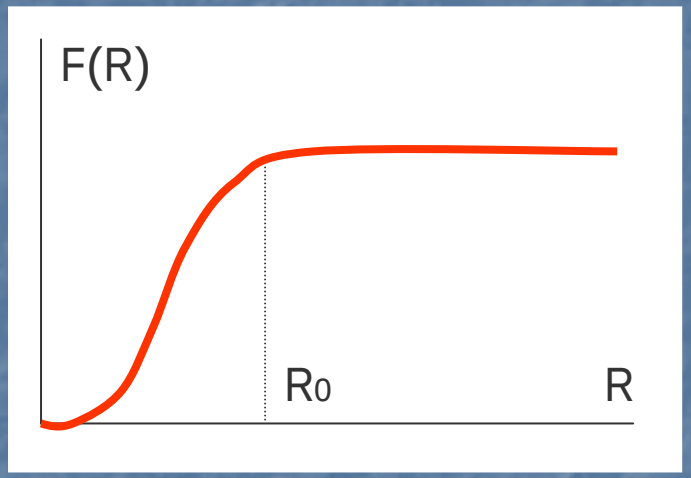


Ideal insulator for sufficiently strong disorder

1D localization is qualitatively different from 3D localization

Repulsion of centers of localization



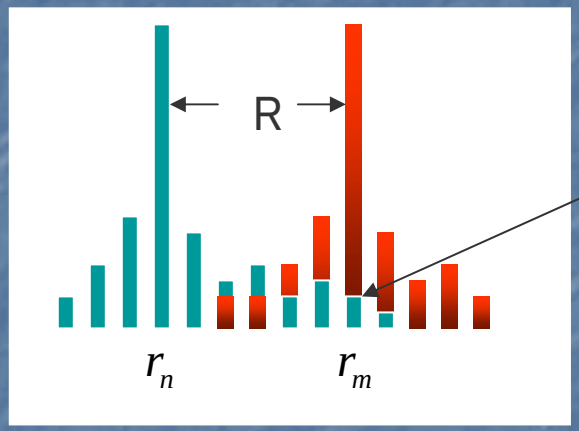


Resonance repulsion of centers of localization

$$R_0 = 2\xi \ln\left(\frac{\delta_\xi}{\omega}\right)$$

$$\omega = \left|E - E'\right| << \delta_\xi$$

Resonance enhancement of overlap



$$|\Psi_n(r_m)|^2 \approx \frac{|H_{nm}|^2}{(E_n - E_m)^2} \sim \left(\right) \exp\left[-\frac{R}{\xi}\right]$$

Enhancement of overlap at $\delta_\xi >> \omega$

$$NC(\omega) \sim \left(\frac{\delta_\xi}{\omega}\right)^2 \int_{R_0}^{\infty} dR \; R^{d-1} \exp\left[-\frac{R}{\xi}\right]$$

$$R_0 = 2\xi \ln\left(\frac{\delta_\xi}{\omega}\right) >> \xi$$

$$NC(\omega) \propto \ln^{d-1}\left(\frac{\delta_\xi}{\omega}\right)$$

An effect similar to Mott's law in the frequency-dependent conductivity

$$\sigma(\omega) \sim \int_{R_0}^{\infty} dR \;\; R^{d-1} R^2 \;\; \exp\left[-\frac{R}{\xi}\right] \propto \omega^2 \ln^{d+1}\left(\frac{\delta_\xi}{|\omega|}\right)$$

At d=1 repulsion of centers of localization and the resonance enhancement of overlap compensate each other

$$NC(\omega) = 1$$

At d>1 resonance enhancement prevails

$$NC(\omega) \propto \ln^{d-1}\left(\frac{\delta_{\xi}}{\omega}\right) >> 1$$

Averaged matrix elements of interaction are enhanced

Summary

- Multifractality of critical eigenfunctions
- Persistence of multifractal texture in a metal and in an insulator phase
- Critical power law and Chalker's scaling
- Critical enhancement of eigenfunction correlations at small energy separations
- Eigenfunction mutual avoiding at large energy separations
- Stratification of coordinate space
- Logarithmic enhancement of correlations in 2D and 3D insulators

Random matrix theories

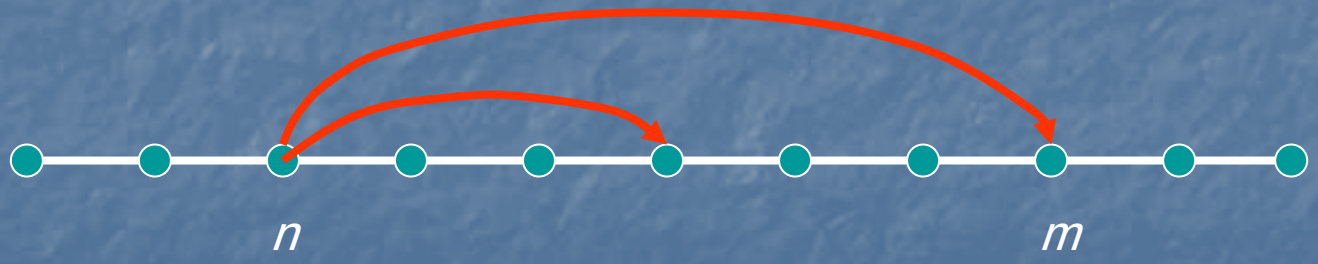
Ideal extended states: classic Wigner-Dyson RMT

$$H_{nm} = H_{mn}^{+}$$

$$\langle H_{nm} \rangle = 0$$

$$\left\langle \left| H_{nm} \right|^2 \right\rangle = 1$$

Independently fluctuating Gaussian random entries



Ideal localized states: random diagonal matrix

$$H_{nm} = \begin{pmatrix} \varepsilon_1 & & & & & \\ & \varepsilon_2 & & & & \\ & & \dots & & & \\ & & & \dots & & \\ & & & & \dots & \\ & & & & & \varepsilon_{N-1} \\ & & & & & & \varepsilon_N \end{pmatrix}$$

$$\Psi_n(r) = \delta_{r,n}$$

Random matrix ensembles with multifractal eigenstates: critical statistics

$$\left\langle \left| H_{nm} \right|^2 \right\rangle = \frac{1}{1+\left(\frac{n-m}{b} \right)^2}$$

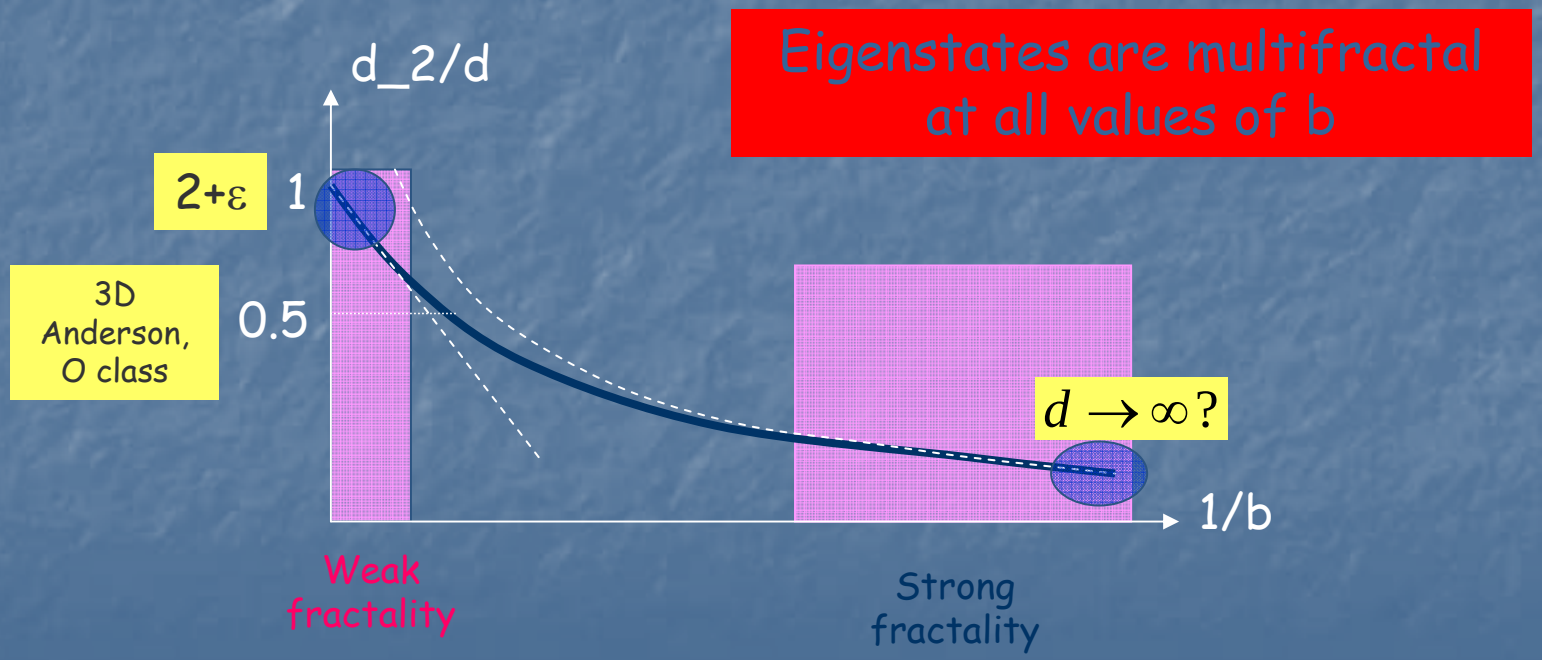
criticality

Controls fractal dimensions

$b \to \infty$ *Wigner-Dyson RMT*

$b \to 0$ *Diagonal RM*

Anderson transition and multifractality at higher dimensions



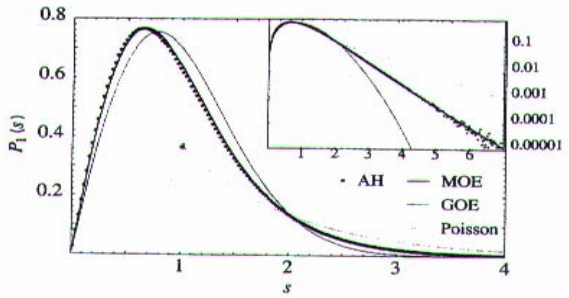
Signature of multifractality

$$P_q = \sum_r \left\langle \left| \Psi_n(r) \right|^{2q} \right\rangle \propto \frac{1}{N^{d_q(q-1)}}$$

$$C(\omega) = \sum_r \left\langle \left| \Psi_n(r) \right|^q \left| \Psi_m(r) \right|^q \delta(E_n - E_m + \omega) \right\rangle \propto \left(\frac{1}{\omega} \right)^{\mu_q}$$

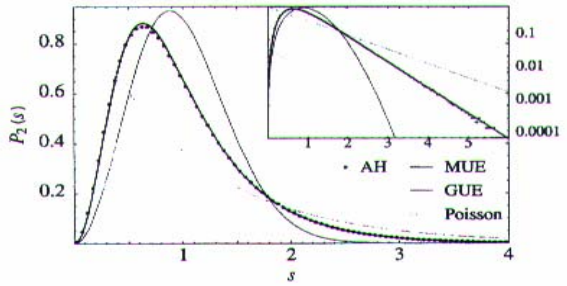
$$\mu_q = 1 - d_q(q-1)/d$$

Spectral statistics



Potential disorder

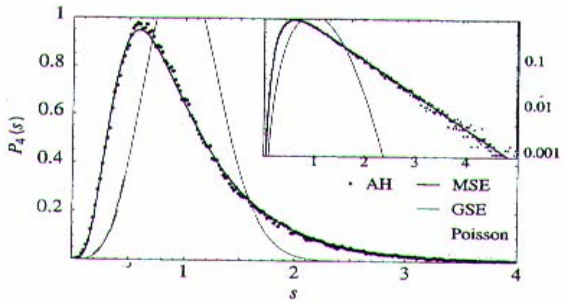
b=0.38



Potential disorder

+ magnetic field

b=0.16

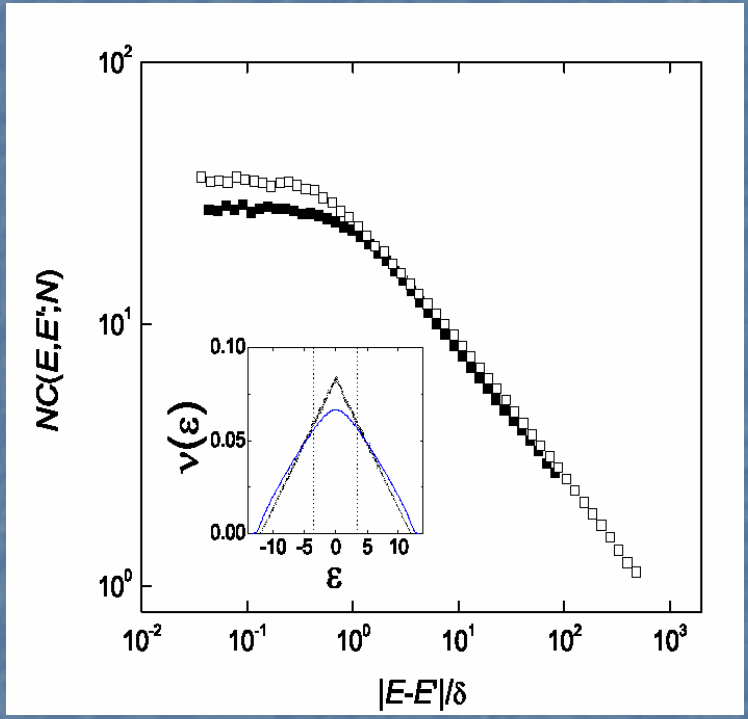


Potential disorder

+ spin-orbit inter.

b=0.07

Critical Eigenfunction correlation: 3D Anderson model vs. RMT



Mobility edge: b=0.42, potential disorder

Random-matrix theory for 3D multifractal insulator

$$\left\langle \left| H_{nm} \right|^2 \right\rangle = \frac{1}{\left(1 + \frac{|n-m|^2}{b^2} \right)} \exp\left[-\left(\frac{|n-m|}{B} \right)^{1/3} \right]$$

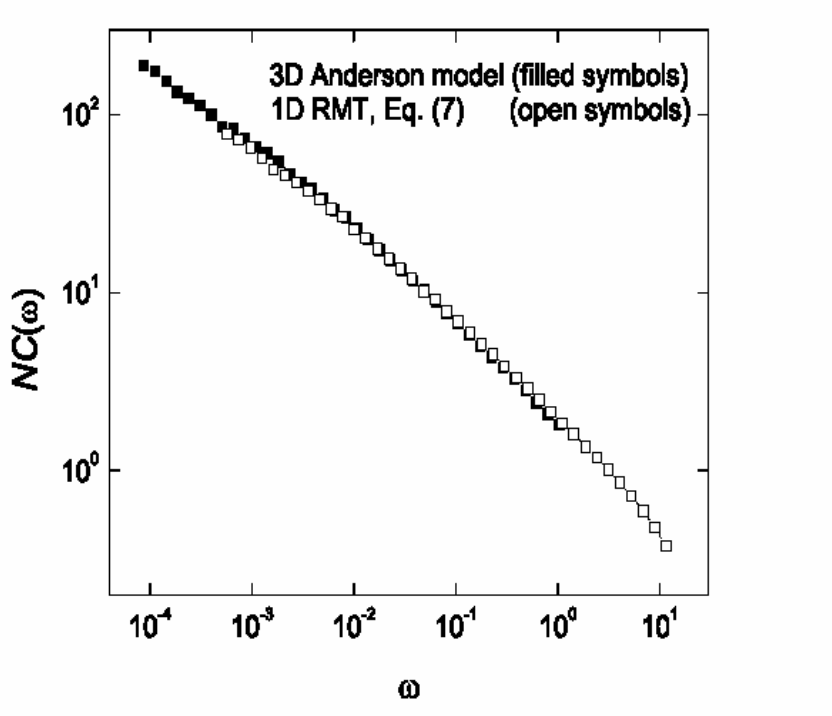
b controls fractality

B controls localization radius

criticality

dimensionality of space

Multifractal insulator: RMT vs. 3D Anderson model



B=5, b=0.42

Possible RMT for a multifractal metal

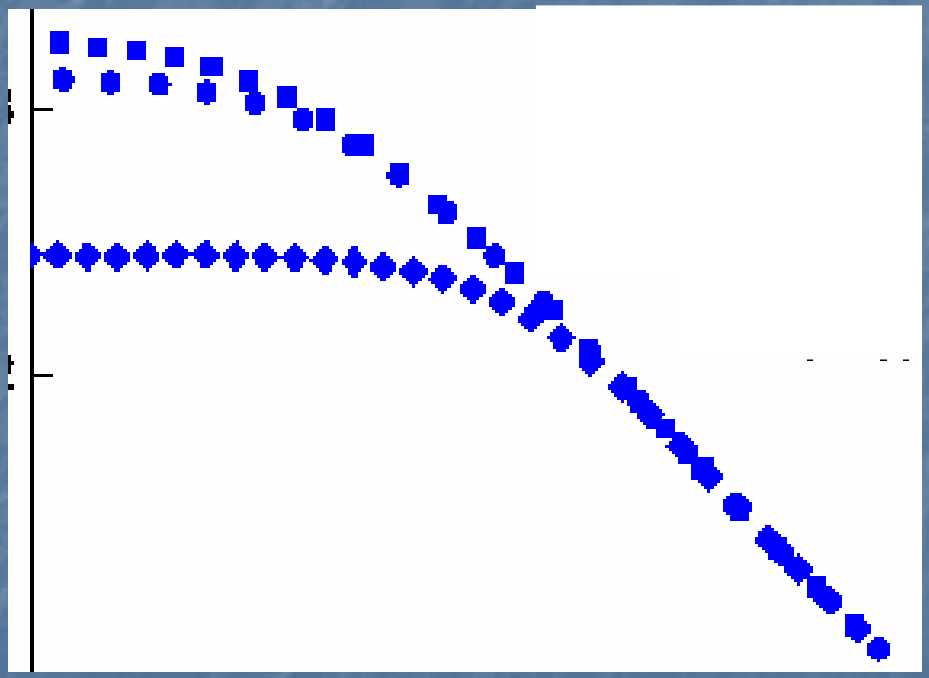
$$\left\langle \left| H_{nm} \right|^2 \right\rangle = \frac{1}{1+\left(\frac{n-m}{b} \right)^{2\alpha}}$$

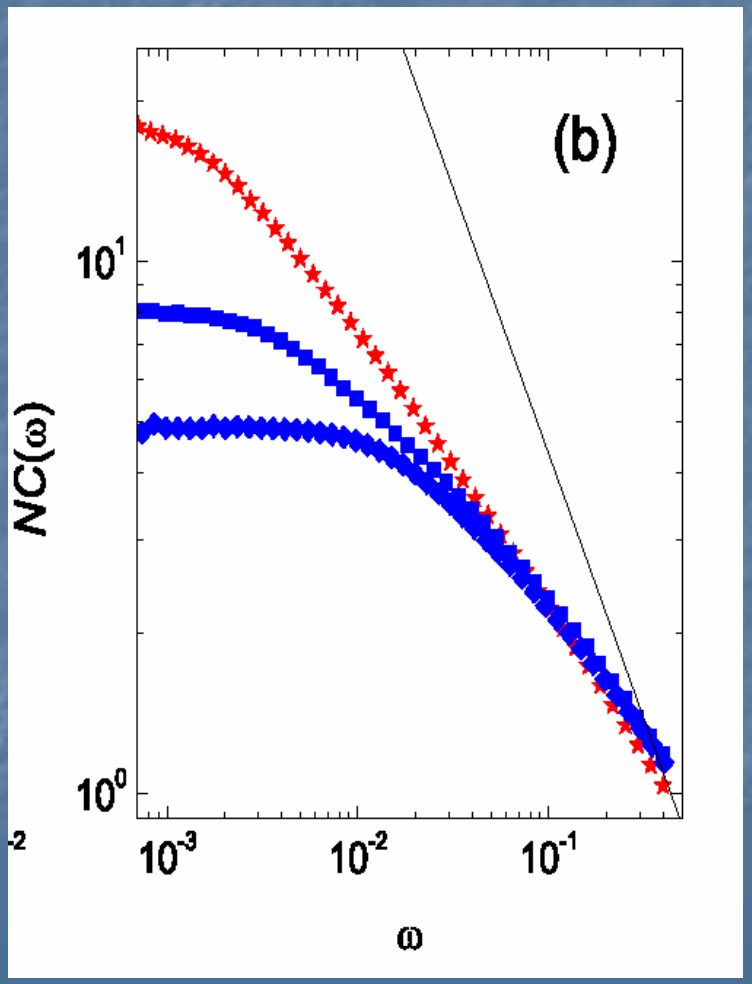
$$\alpha < 1$$

$$\xi \propto \exp\left[\frac{1}{1-\alpha} \right]$$

α marks departure from criticality

Multifractal metal: 3D Anderson vs RMT





Conclusion

- Random matrix models for ideal extended and localized states
- Critical random matrix model with multifractal eigenstates
- Random matrix models for a multifractal metal and an insulator: good description of the off-critical states

Those RMT are generators of non-trivial single-particle eigenststes to be used as a basis to treat electron interaction.

Cooper instability near the Anderson transition

Anderson vs Anderson

Anderson theorem: *Tc does not depend on concentration of nonmagnetic impurities*

$$\Delta(r) = \int dr'\Delta(r')K(r,r';T)$$

$$K(r,r';T) = \lambda \sum_{ij} \eta_{ij}(T)$$

$$\eta_{ij}(T) = \frac{\tanh(E_i/2T) + \tanh(E_j/2T)}{E_i + E_j}$$

If $\Delta(r)$ does not depend on r and $\Psi(r)$ is real then the properties of eigenfunctions does not enter due to the normalization condition

$$\int dr\Psi_n(r)\Psi_m^*(r) = \delta_{nm}$$

For strong disorder ANDERSON THEOREM FAILS

What to do when $\Delta(r)$ significantly depends on r ?

At T=Tc the operator K acquires the eigenvalue 1

$$Tr\frac{1}{1-K} = Tr(1+K+K^2+...+K^n+...) = \infty$$

$$TrK^{n+1} = TrK^{n}, \quad as \quad n \to \infty$$

$$TrK^{'}TrK^{2} = \sum_{ij}\eta_{ij}\eta_{kl}\sum_{r,r_1}\int dr dr_1 \Psi_i(r)\Psi_j(r)\Psi_j^*(r_1)\Psi_i^*(r_1)\ \Psi_k(r_1)\Psi_l(r_1)\Psi_l^*(r)\Psi_k^*(r)$$

Neglect off-diagonal terms with i,j,k,l all different

Retain only diagonal elements $\langle\Psi_i^2(r)\Psi_j^2(r)\rangle$ *and the terms with maximal number of summations*

The new MF equation:

$$\Delta_i = \lambda \sum_j \eta_j M_{ij} \Delta_j$$

$$M_{ij} = \left\langle \sum_r \Psi_i^2(r) \Psi_j^2(r) \right\rangle \propto \frac{(\omega / E_0)^{d_2/d-1}}{N}$$

$$\eta_i = \frac{\tanh(E_i / 2T)}{E_i}$$

$$\Rightarrow \Delta(\varepsilon) = \lambda \int \frac{\tanh(\varepsilon'/2T)}{\varepsilon'} \tilde{K}(\varepsilon - \varepsilon')\Delta(\varepsilon')d\varepsilon'$$

$$\tilde{K}(\omega) = NC(\omega) = \left(\frac{E_0}{\omega} \right)^{1-d_2/d}$$

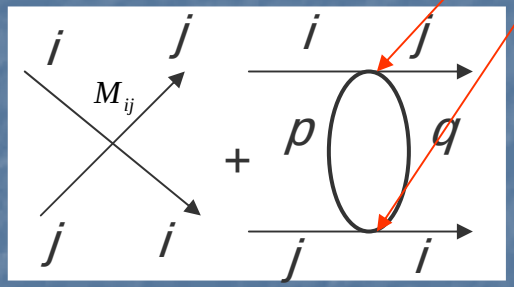
How good is the approximation?

Neglected:

$$M_{ijkl}^2 = \left\langle \left(\sum_r \Psi_i(r)\Psi_j(r)\Psi_k(r)\Psi_l(r) \right)^2 \right\rangle \propto \frac{(\omega/E_0)^{3d_4/d-1}}{N^3}$$

The true small parameter:

$$\frac{\lambda M_{ijkl}^2 N^2}{\omega M_{ij}} = \left(\frac{\omega}{E_0} \right)^{\frac{3d_4-2d_2}{d}}$$



$$\sum_r |\Psi_n(r)|^8 = \frac{1}{N^{3d_4}} < \left(\sum_r |\Psi_n(r)|^4 \right)^2 = \frac{1}{N^{2d_2}}$$

$$3d_4 - 2d_2 > 0$$

3D Anderson:

$$3d_4 - 2d_2 = 0.5$$

Solution to the MF equation

$$\Rightarrow \Delta(\varepsilon) = \lambda \int \frac{\tanh(\varepsilon'/2T)}{\varepsilon'} \left(\frac{E_0}{|\varepsilon - \varepsilon'|} \right)^{1-d_2/d} \Delta(\varepsilon') d\varepsilon'$$

$$\Delta(\varepsilon) = f(\varepsilon/2T)$$

$$\varepsilon, \varepsilon' \to 2Tx, \quad 2Ty$$

λ will enter in the combination $\tilde{\lambda} = \lambda \left(\frac{E_0}{2T} \right)^{1-d_2/d}$

First non-trivial solution corresponds to $\lambda_c \sim 1$

$$T_c = cE_0 \lambda^{1/\mu}, \qquad \mu = 1 - d_2/d.$$

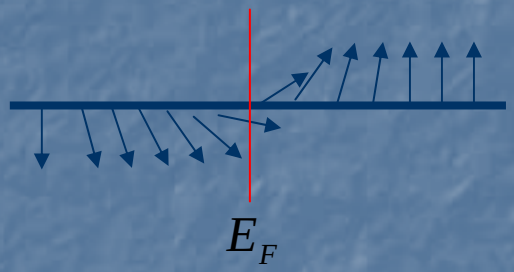
How robust in the MF result?

Other approaches

Anderson spin representation of superconducting Hamiltonian

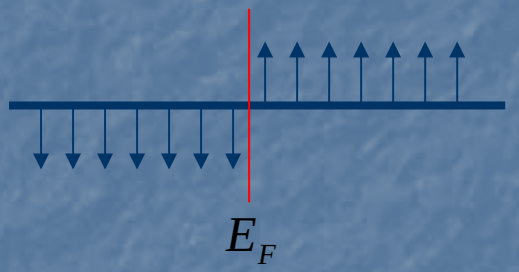
$$H_{eff} = -2\sum_i \varepsilon_i S_i^{\,z} - \sum_{i\neq j} M_{ij}\left(S_i^{\,x} S_j^{\,x} + S_i^{\,y} S_j^{\,y}\right)$$

Off-diagonal matrix elements $M_{ijkl} = \sum_r \left\langle \Psi_i(r)\Psi_j(r)\Psi_k(r)\Psi_l(r) \right\rangle$ are neglected



Superconducting phase

$$\langle S_i^{\,x,y} \rangle \neq 0$$



Normal phase

$$\langle S_i^{\,x,y} \rangle = 0$$

Cooper susceptibility

$$\chi(T) = \left\langle \frac{\partial S_i^+}{\partial h} \right\rangle_{h\to 0}$$

$$\delta H = \sum_{i,|\varepsilon_i|<\omega_0} h S_i^- + \bar{h} S_i^+$$

$$\chi(T) = \chi_1(T) + \chi_2(T) + \chi_3(T) + \ldots$$

$$\chi_1(T) = \frac{1}{N} \sum_i^{|\varepsilon_i|<\omega_0} \frac{\tanh\left(\frac{\varepsilon_i}{T}\right)}{2\varepsilon_i}$$

$$\chi_2(T) = \frac{1}{4N} \sum_{i\neq j}^{|\varepsilon_{i,j}|<\omega_0} \tanh\left(\frac{\tilde{\varepsilon}_i}{T}\right) \tanh\left(\frac{\tilde{\varepsilon}_j}{T}\right) \frac{\quad}{\tilde{\varepsilon}_i \tilde{\varepsilon}_j}$$

$$\tilde{\varepsilon}_{i,j} = \frac{\varepsilon_i + \varepsilon_j}{2} \pm \sqrt{\frac{(\varepsilon_i - \varepsilon_j)^2}{4} + \quad}$$

Superconducting transition temperature

$$\chi(T_c) = \sum_n \chi_n(T_c) = \infty$$

Exact! $\Longrightarrow$ $$\lim_{n\to\infty} \frac{\chi_{n+1}(T_c)}{\chi_n(T_c)} = 1$$

Replaced by: $\chi_1(T_c^{(0)}) = \chi_2(T_c^{(0)})$ ***OR*** $\chi_2(T_c^{(1)}) = \chi_3(T_c^{(1)})$

Operational definitions of Tc for numerical simulations

MFA vs virial expansion

Virial: $T^0_c(\lambda,\gamma)=2.1\lambda^{1.79\pm0.05}$

MFA: $T^0_c(\lambda,\gamma)=2.46\lambda^{1.78}$

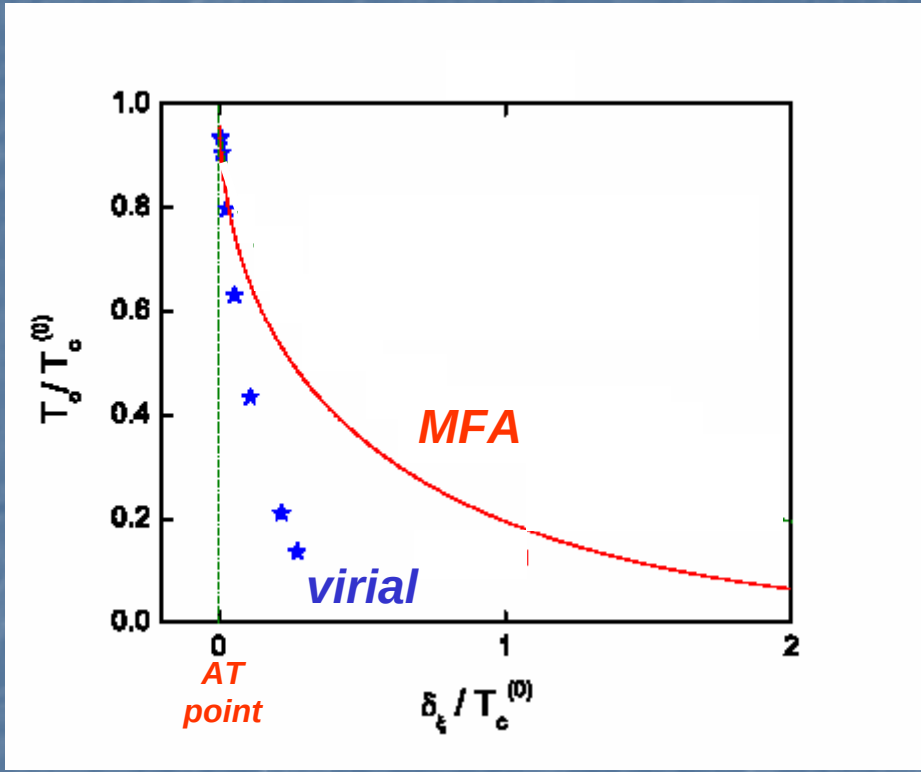
$$\frac{1}{1-d_2/d}=1.78$$

MFA: neglecting thermal fluctuations and non-local spacial fluctuations

Virial expansion: neglecting higher-order terms of virial expansion

Good agreement of results of different approximations

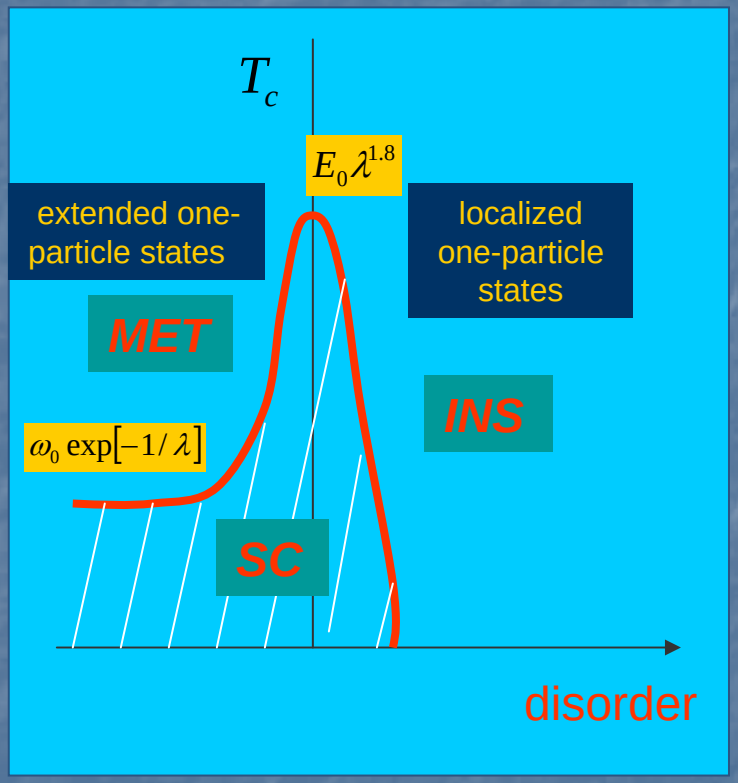
MF vs Virial in the insulator



$$\delta_\xi = \frac{1}{\rho \xi^d}$$

Virial expansion is probably closer to Tc of global phase coherence

Enhancement of T_c near the Anderson transition

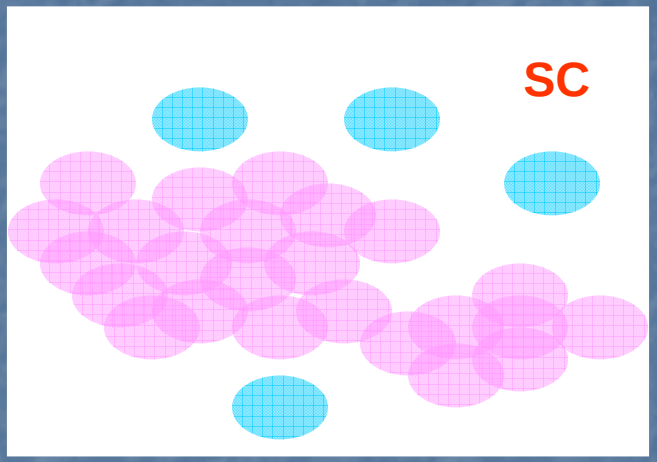


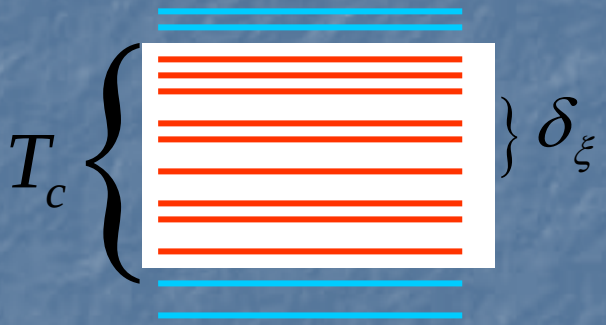
No Coulomb interaction

Possibly realizable in cold atoms in imperfect optical traps

Superconductor-Insulator transition: percolation without granulation

$T_c(crit) > \delta_\xi$





Only states in the strip ~T_c near the Fermi level take part in superconductivity

$T_c(crit) < \delta_\xi$