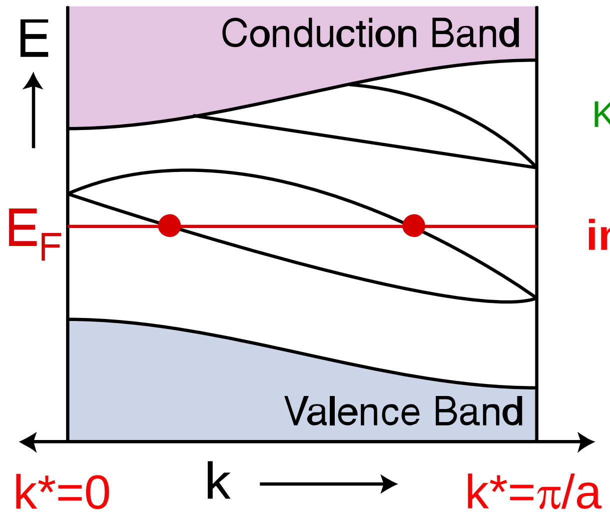
Time Reversal Invariant Z_2 Topological Insulator

2D Bloch Hamiltonians subject to the T constraint $\Theta H(\mathbf{k})\Theta^{-1} = H(-\mathbf{k})$

with $\Theta^2 = -1$ are classified by a Z_2 topological invariant (ν = 0,1)

Understand via Bulk-Boundary correspondence : Edge States for 0<k<π/a

ν=0 : Conventional Insulator

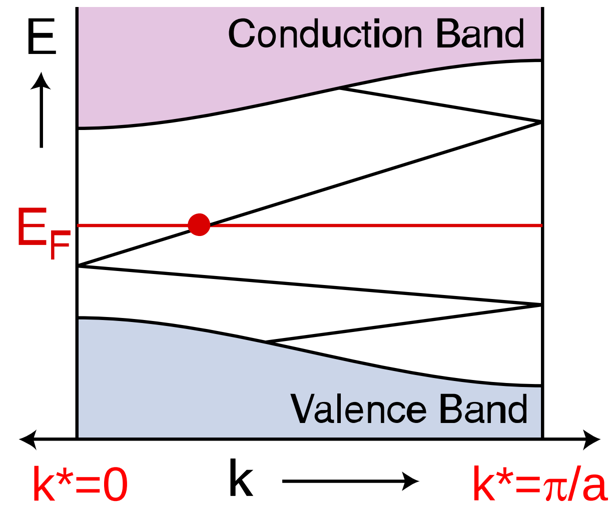


Even number of bands crossing Fermi energy

Kramers degenerate at **time reversal invariant momenta**

k* = –k* + G

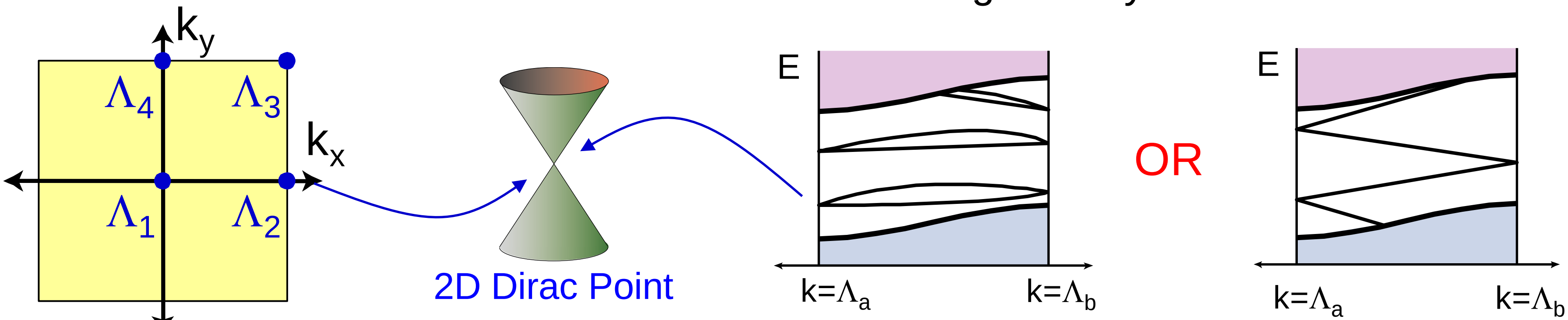
ν=1 : Topological Insulator



Odd number of bands crossing Fermi energy

3D Topological Insulators

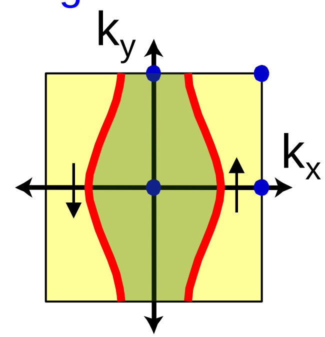
There are 4 surface Dirac Points due to Kramers degeneracy



Surface Brillouin Zone

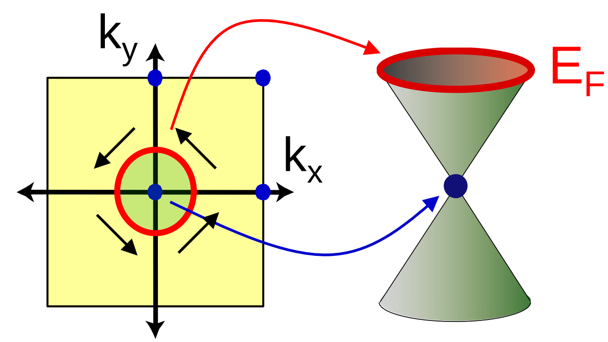
How do the Dirac points connect? Determined by 4 bulk Z_2 topological invariants ν_0 ; $(\nu_1\nu_2\nu_3)$

ν_0 = 0 : Weak Topological Insulator



Related to layered 2D QSHI ; $(\nu_1\nu_2\nu_3)$ ~ Miller indices
Fermi surface encloses even number of Dirac points

ν_0 = 1 : Strong Topological Insulator



Fermi circle encloses odd number of Dirac points

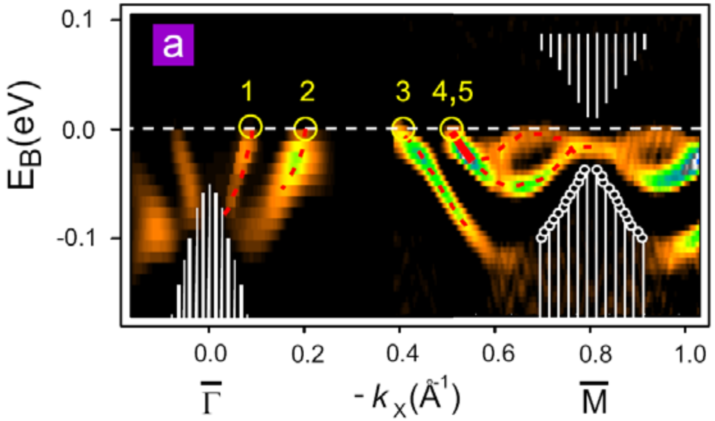
Topological Metal :

- 1/4 graphene
- Berry's phase π
- Robust to disorder: impossible to localize

$Bi_{1-x}Sb_x$

Theory: Predict $Bi_{1-x}Sb_x$ is a topological insulator by exploiting inversion symmetry of pure Bi, Sb (Fu,Kane PRL'07)

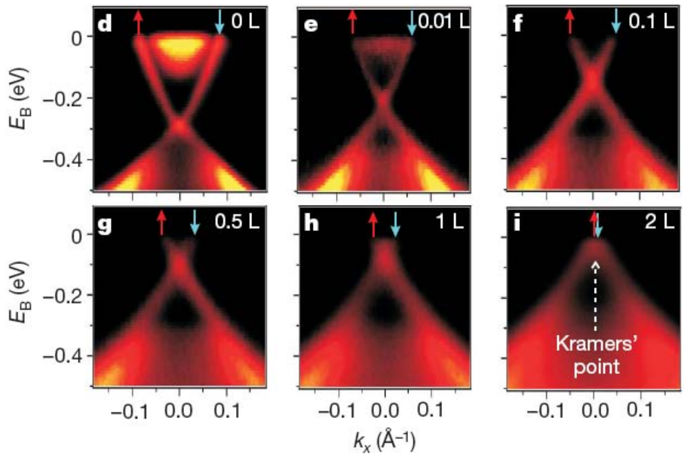
Experiment: ARPES (Hsieh et al. Nature '08)



- $Bi_{1-x}Sb_x$ is a Strong Topological Insulator $\nu_0;(\nu_1,\nu_2,\nu_3) = 1;(111)$
- 5 surface state bands cross E_F between Γ and M

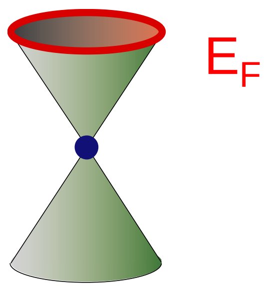
Bi_2Se_3

ARPES Experiment : Y. Xia et al., Nature Phys. (2009).
Band Theory : H. Zhang et. al, Nature Phys. (2009).



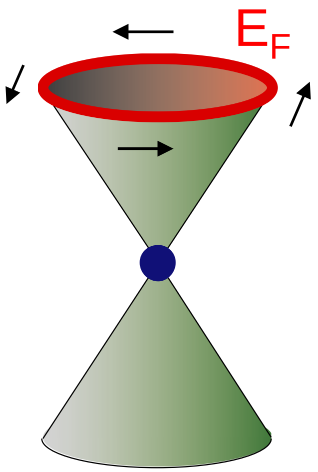
Control E_F on surface by exposing to NO_2

- $\nu_0;(\nu_1,\nu_2,\nu_3) = 1;(000)$: Band inversion at Γ
- Energy gap: $\Delta \sim .3$ eV : A room temperature topological insulator
- Simple surface state structure : Similar to graphene, except only a single Dirac point



Unique Properties of Topological Insulator Surface States

"Half" an ordinary 2DEG ; ¼ Graphene



Spin polarized Fermi surface

- Charge Current ~ Spin Density
- Spin Current ~ Charge Density

π Berry's phase

- Robust to disorder
- Weak Antilocalization
- Impossible to localize, Klein paradox

Exotic States when broken symmetry leads to surface energy gap:

- Quantum Hall state, topological magnetoelectric effect

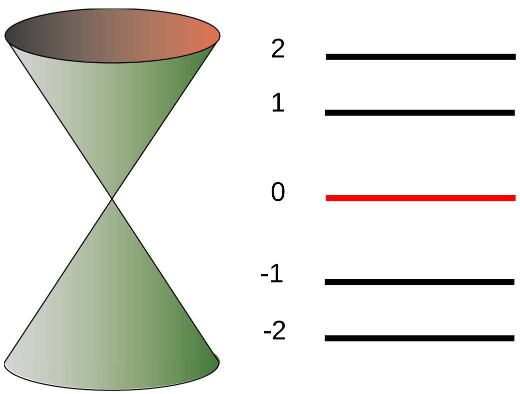
  Fu, Kane '07; Qi, Hughes, Zhang '08, Essin, Moore, Vanderbilt '09

- Superconducting state

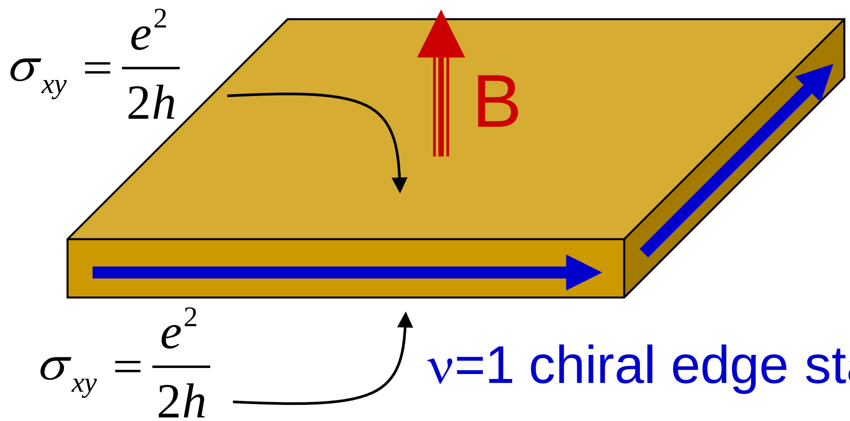
  Fu, Kane '08

Surface Quantum Hall Effect

Orbital QHE : E=0 Landau Level for Dirac fermions. “Fractional” IQHE



$$\sigma_{xy} = \frac{e^2}{h}\left(n + \frac{1}{2}\right)$$

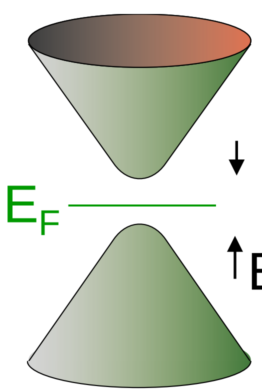


ν=1 chiral edge state

Anomalous QHE : Induce a surface gap by depositing magnetic material

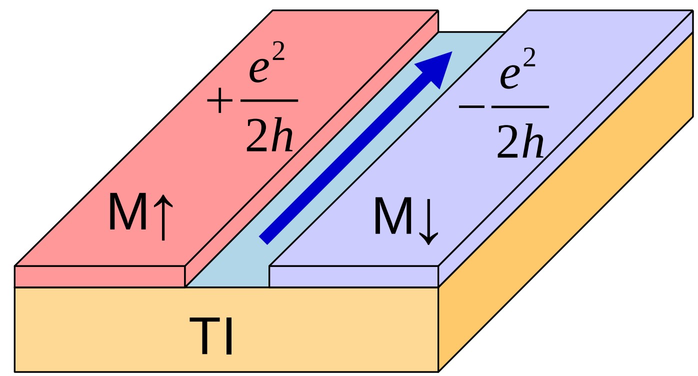
$$H_0 = \psi^\dagger(-iv\vec{\sigma}\cdot\vec{\nabla} - \mu + \Delta_M \sigma_z)\psi$$

Mass due to Exchange field



$E_{gap} = 2|\Delta_M|$

$$\sigma_{xy} = \mathrm{sgn}(\Delta_M)\frac{e^2}{2h}$$



Chiral Edge State at Domain Wall : $\Delta_M \leftrightarrow -\Delta_M$

Topological Superconductors, Majorana Fermions and Topological Quantum Computation

1. Bogoliubov de Gennes Theory
2. Majorana bound states, Kitaev model
3. Topological superconductor
4. Periodic Table of topological insulators and superconductors
5. Topological quantum computation
6. Proximity effect devices

BCS Theory of Superconductivity

mean field theory : $\Psi^\dagger\Psi\Psi^\dagger\Psi \Rightarrow \langle\Psi^\dagger\Psi^\dagger\rangle\Psi\Psi = \Delta^*\Psi\Psi$

$$H = \sum_{\mathbf{k}} \begin{pmatrix}\Psi^\dagger & \Psi\end{pmatrix} H_{BdG} \begin{pmatrix}\Psi \\ \Psi^\dagger\end{pmatrix}$$

Bogoliubov de Gennes Hamiltonian

$$H_{BdG} = \begin{pmatrix} H_0 & \Delta \\ \Delta^* & -H_0 \end{pmatrix}$$

Intrinsic anti-unitary particle – hole symmetry

$$\Xi H_{BdG} \Xi^{-1} = -H_{BdG}$$

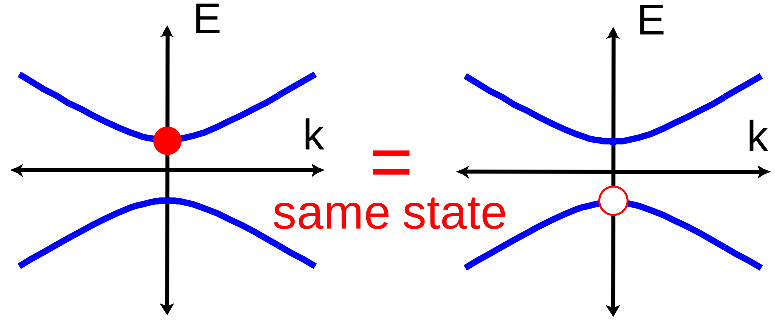
$$\Xi\varphi = \tau_x \varphi^*$$

$$\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Xi^2 = +1$$

Particle – hole redundancy

$$\varphi_{-E} = \Xi\varphi_E \Rightarrow \gamma_E^\dagger = \gamma_{-E}$$



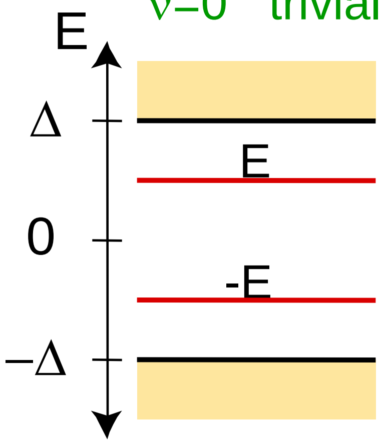
Bloch - BdG Hamiltonians satisfy $\Xi H_{BdG}(\mathbf{k})\Xi^{-1} = -H_{BdG}(-\mathbf{k})$

Topological classification problem similar to time reversal symmetry

1D Z_2 Topological Superconductor : $\nu = 0,1$ (Kitaev, 2000)

Bulk-Boundary correspondence : Discrete end state spectrum

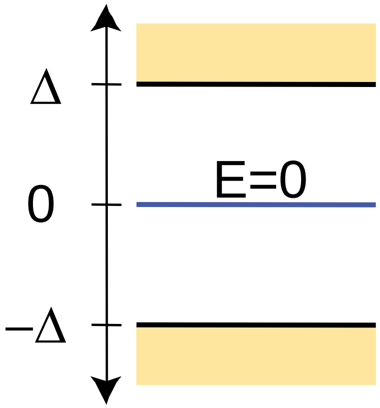
ν=0 "trivial"



$$\Gamma_E$$

$$\Gamma_{-E} = \Gamma_E^\dagger$$

ν=1 "topological"



Zero mode

$$\Gamma_{E=0}^\dagger = \Gamma_{E=0} \equiv \gamma$$

Majorana fermion bound state

Majorana Fermion : Particle = Antiparticle $\gamma = \gamma^\dagger$

Real part of a Dirac fermion :

$$\begin{cases} \gamma_1 = \Psi + \Psi^\dagger & ; \quad \Psi = \gamma_1 + i\gamma_2 \\ \gamma_2 = -i(\Psi - \Psi^\dagger) & ; \quad \Psi^\dagger = \gamma_1 - i\gamma_2 \end{cases}$$

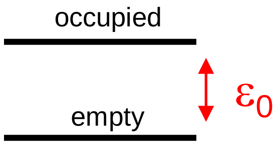
$$\gamma_i^2 = 1$$

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

"Half a state"

Two Majorana fermions define a single two level system

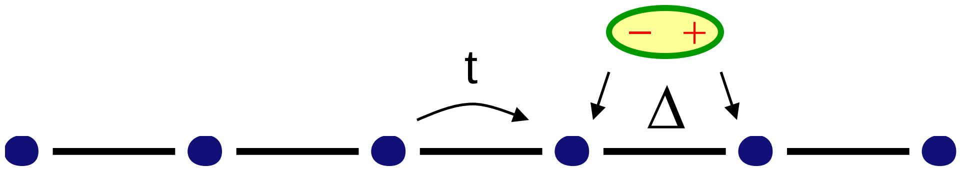
$$H = 2i\varepsilon_0\gamma_1\gamma_2 = \varepsilon_0\Psi^\dagger\Psi$$



Kitaev Model for 1D p wave superconductor

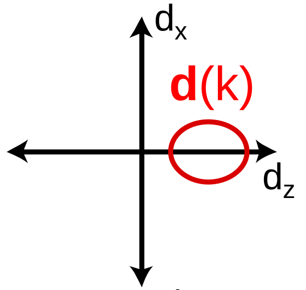
$$H - \mu N = \sum_i t(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) - \mu c_i^\dagger c_i + \Delta(c_i c_{i+1} + c_{i+1}^\dagger c_i^\dagger)$$

$$= \sum_k \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} H_{BdG}(k) \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}$$

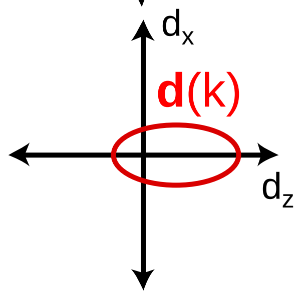


$$H_{BdG}(k) = \tau_z(2t\cos k - \mu) + \tau_x \Delta \sin k = \mathbf{d}(k)\cdot\vec{\tau}$$

$|\mu|>2t$: Strong pairing phase
trivial superconductor



$|\mu|<2t$: Weak pairing phase
topological superconductor



Similar to SSH model, except different symmetry : $(d_x, d_y, d_z)\big|_k = (-d_x, -d_y, d_z)\big|_{-k}$

Majorana Chain

$$c_i \to \gamma_{1i} + i\gamma_{2i}$$

$$H = 2i\sum_i t_1\gamma_{1i}\gamma_{2i} + t_2\gamma_{2i}\gamma_{1i+1}$$

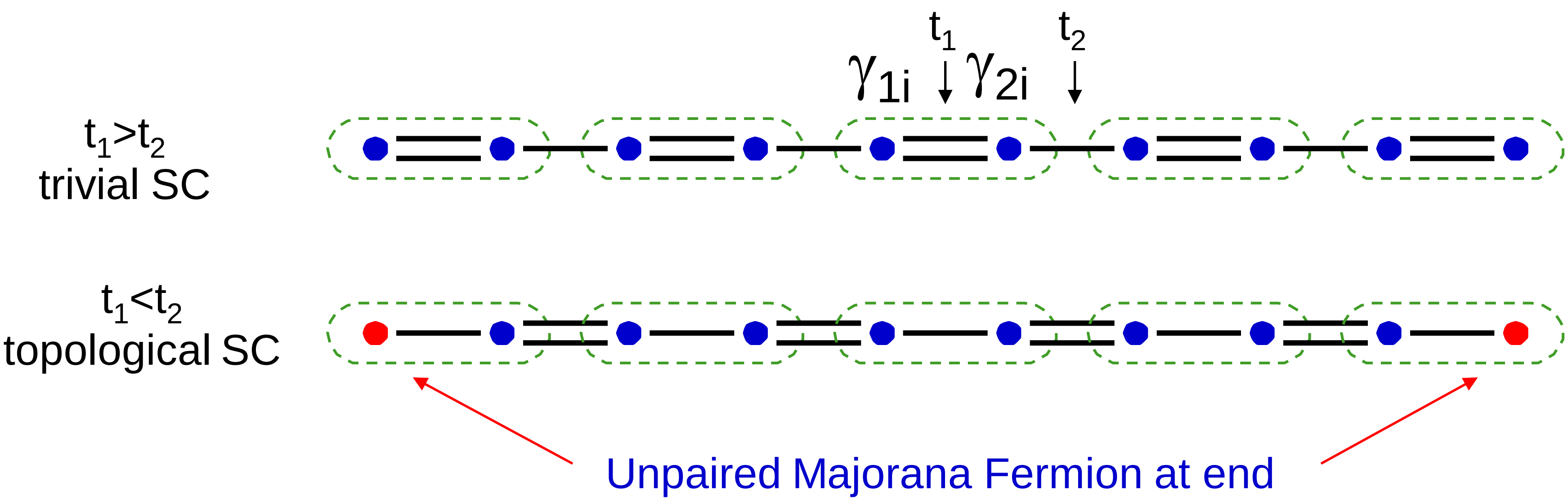
$$\mu c_i^\dagger c_i \to 2i\mu\gamma_{1i}\gamma_{2i}$$

$$t\left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i\right) \to 2it\left(\gamma_{1i}\gamma_{2i+1} - \gamma_{2i}\gamma_{1i+1}\right)$$

$$\Delta\left(c_i c_{i+1} + c_{i+1}^\dagger c_i^\dagger\right) \to 2i\Delta\left(\gamma_{1i}\gamma_{2i+1} + \gamma_{2i}\gamma_{1i+1}\right)$$

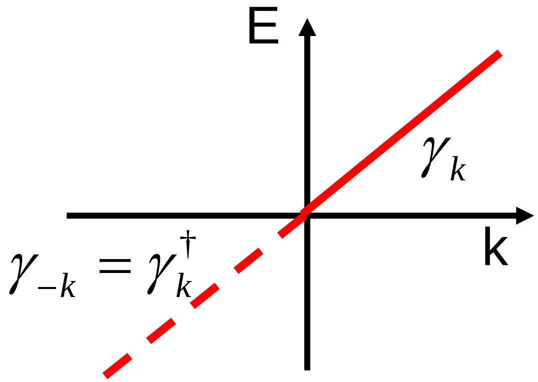
For Δ=t : nearest neighbor Majorana chain

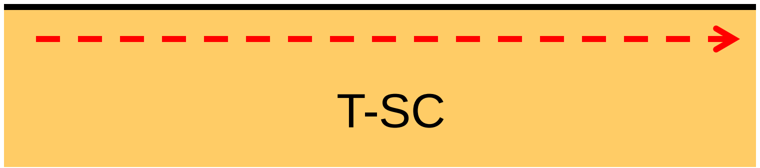
$$t_1 = \mu,\ \ t_2 = 2t$$



2D $\mathbb{Z}$ topological superconductor (broken T symmetry)

Bulk-Boundary correspondence: n = # Chiral Majorana Fermion edge states





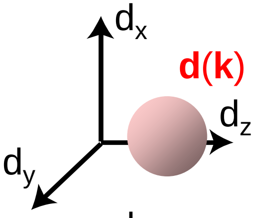
Examples

- Spinless p_x+ip_y superconductor (n=1)
- Chiral triplet p wave superconductor (eg Sr_2RuO_4) (n=2)

Read Green model : $H = \sum_{\mathbf{k}} \left(\frac{\mathbf{k}^2}{2m} - \mu \right) c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + \left(\Delta(\mathbf{k}) c_{\mathbf{k}} c_{-\mathbf{k}} + c.c. \right)$ $\Delta(\mathbf{k}) = \Delta_0 \left(k_x + i k_y \right)$

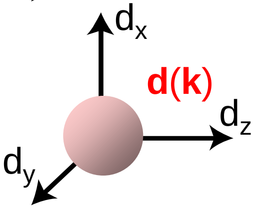
Lattice BdG model : $H_{BdG}(\mathbf{k}) = \tau_z \left(2t \left[\cos k_x + \cos k_y \right] - \mu \right) + \Delta \left(\tau_x \sin k_x + \tau_y \sin k_y \right) = \mathbf{d}(k) \cdot \vec{\tau}$

$|\mu|>4t$: Strong pairing phase
trivial superconductor



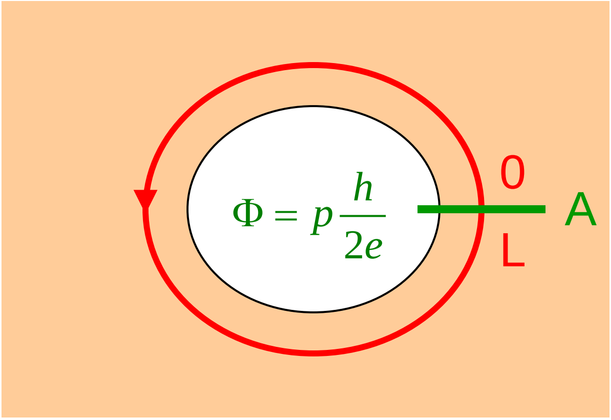
Chern number 0

$|\mu|<4t$: Weak pairing phase
topological superconductor



Chern number 1

Majorana zero mode at a vortex

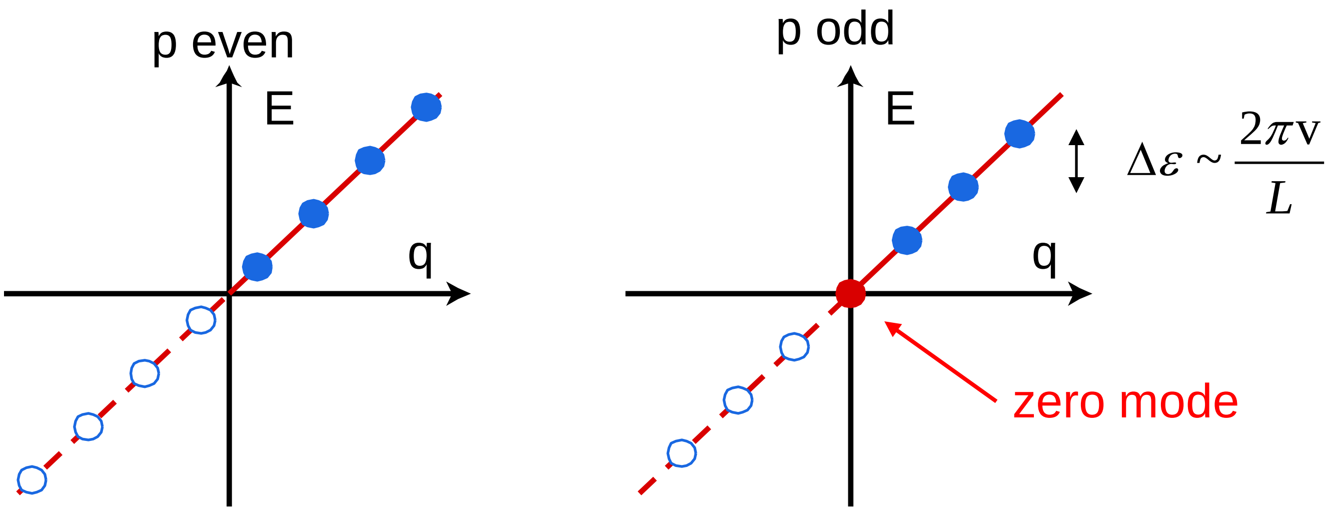


Hole in a topological superconductor threaded by flux

Boundary condition on fermion wavefunction

$$\psi(L) = (-1)^{p+1}\psi(0)$$

$$\psi(x) \propto e^{iq_m x} \quad ; \quad q_m = \frac{\pi}{L}(2m+1+p)$$



Without the hole : Caroli, de Gennes, Matricon theory ('64) $\Delta\varepsilon \sim \frac{\Delta^2}{E_F}$

Majorana Fermions and Topological Quantum Computing

(Kitaev '03)

The degenerate states associated with Majorana zero modes define a topologically protected quantum memory

- 2 Majorana separated bound states = 1 fermion $\Psi = \gamma_1 + i\gamma_2$
  - 2 degenerate states (full/empty) = 1 qubit
- 2N separated Majoranas = N qubits
- Quantum Information is stored non locally
  - Immune from local decoherence

Braiding performs unitary operations

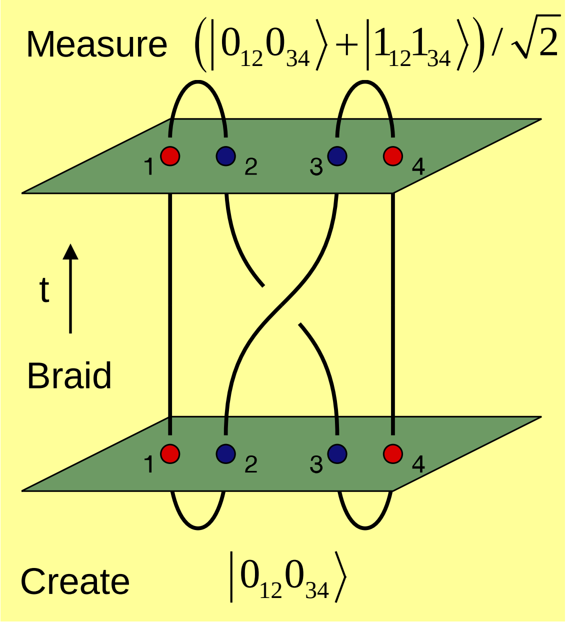
Non-Abelian statistics

Interchange rule (Ivanov 03)

$$\gamma_i \to \gamma_j$$

$$\gamma_j \to -\gamma_i$$

These operations, however, are not sufficient to make a universal quantum computer



Potential condensed matter hosts for Majorana bound states

- Quasiparticles in fractional Quantum Hall effect at ν=5/2 Moore Read '91

- Unconventional superconductors

  - Sr_2RuO_4 Das Sarma, Nayak, Tewari '06
  - Fermionic atoms near feshbach resonance Gurarie '05

- Proximity Effect Devices using ordinary s wave superconductors

  - Topological Insulator devices Fu, Kane '08
  - Semiconductor/Magnet devices Sau, Lutchyn, Tewari, Das Sarma '09, Lee '09, ...

- .... among others

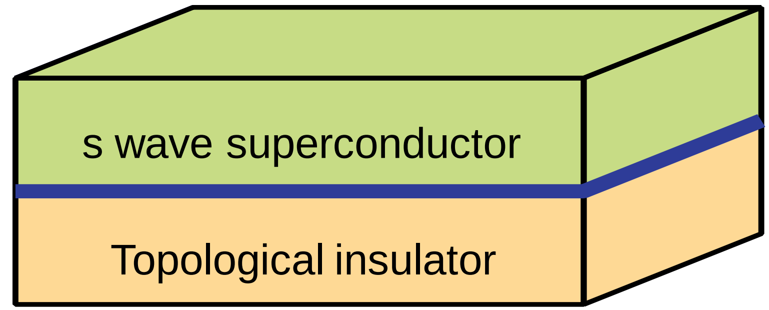
Current Status : Not Observed

Superconducting Proximity Effect

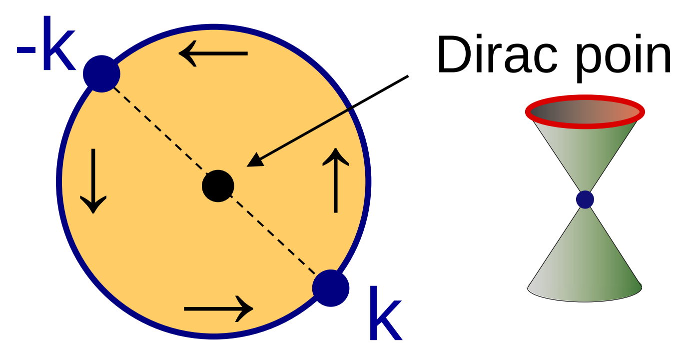
$$H = \psi^{\dagger}(-iv\vec{\sigma}\cdot\vec{\nabla} - \mu)\psi$$

$$+\Delta_S \psi_{\uparrow}^{\dagger}\psi_{\downarrow}^{\dagger} + \Delta_S^{*}\psi_{\downarrow}\psi_{\uparrow}$$

proximity induced superconductivity at surface

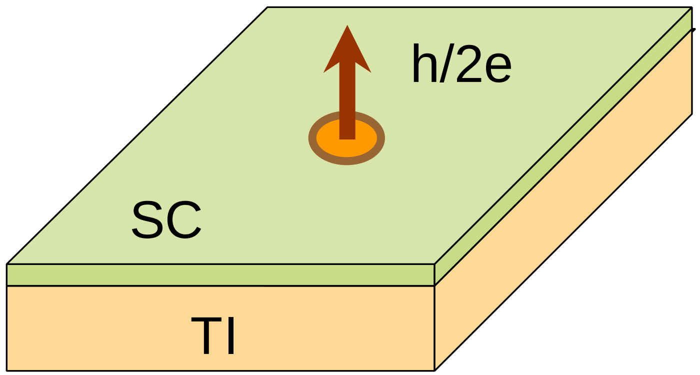


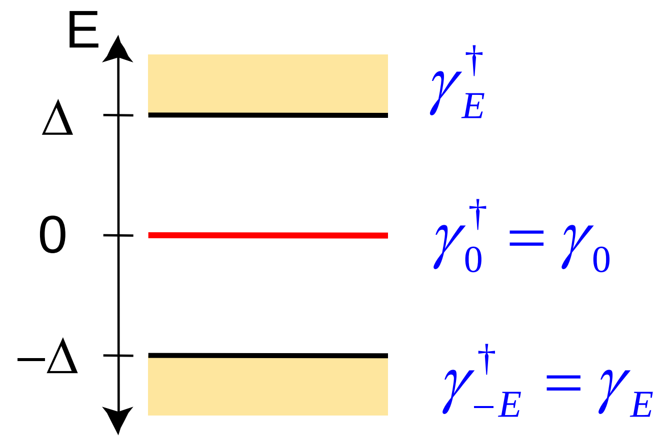
- Half an ordinary superconductor
- Similar to spinless p_x+ip_y superconductor, except :
  - Does not violate time reversal symmetry
  - s-wave singlet superconductivity
  - Required minus sign is provided by π Berry's phase due to Dirac Point
- Nontrivial ground state supports Majorana fermion bound states at vortices



Majorana Bound States on Topological Insulators

1. h/2e vortex in 2D superconducting state

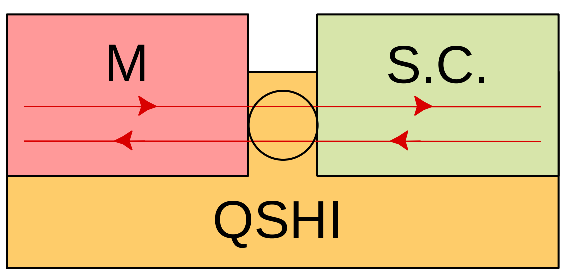




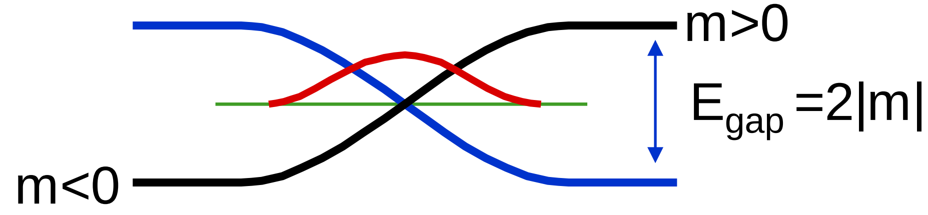
Quasiparticle Bound state at E=0

Majorana Fermion γ_0 "Half a State"

2. Superconductor-magnet interface at edge of 2D QSHI



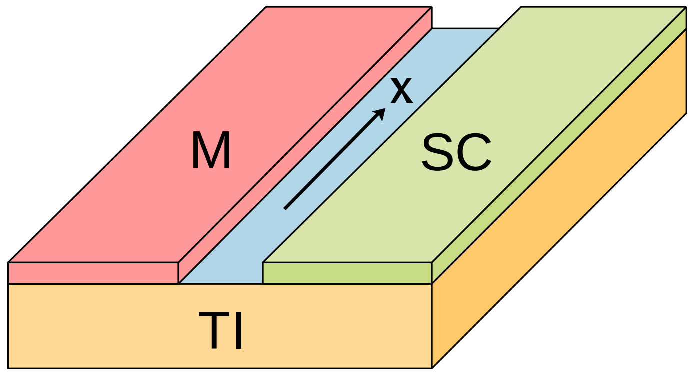
$$m = |\Delta_S| - |\Delta_M|$$

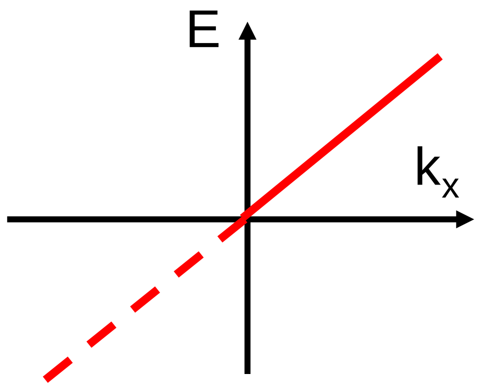


Domain wall bound state γ_0

1D Majorana Fermions on Topological Insulators

1. 1D Chiral Majorana mode at superconductor-magnet interface

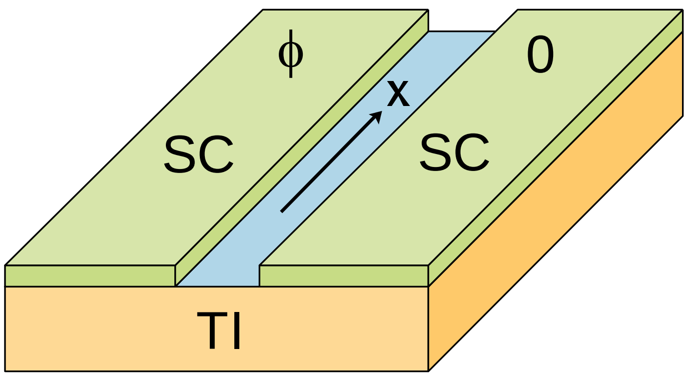


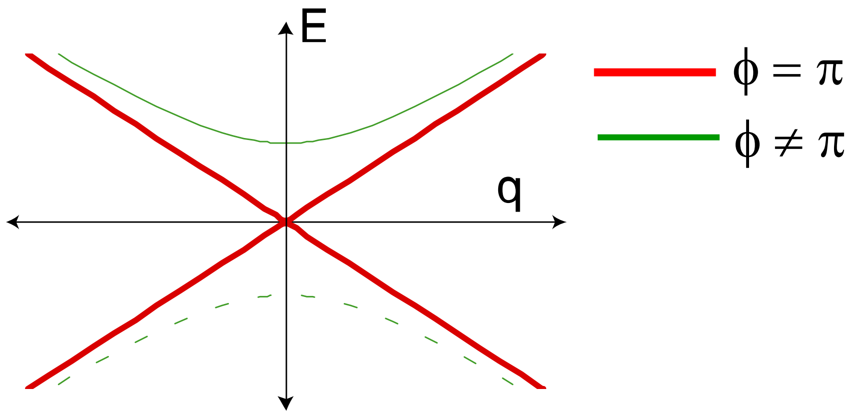


$\gamma_k = \gamma_{-k}^{\dagger}$: "Half" a 1D chiral Dirac fermion

$$H = -i\hbar v_F \; \gamma \partial_x \gamma$$

2. S-TI-S Josephson Junction





Gapless non-chiral Majorana fermion for phase difference $\phi = \pi$

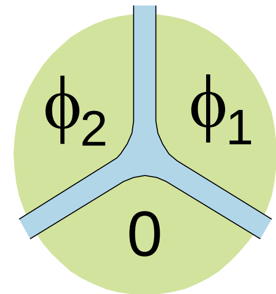
$$H = -i\hbar v_F \; \left(\gamma_L \partial_x \gamma_L - \gamma_R \partial_x \gamma_R\right) + i\Delta \cos(\phi/2)\gamma_L \gamma_R$$

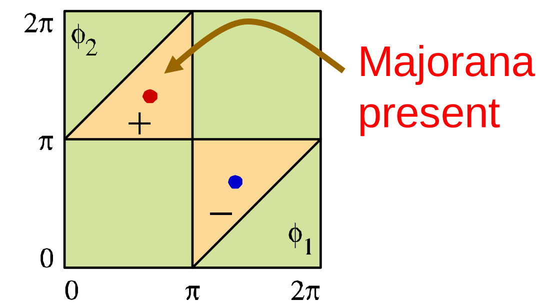
Manipulation of Majorana Fermions

Control phases of S-TI-S Junctions

Tri-Junction :

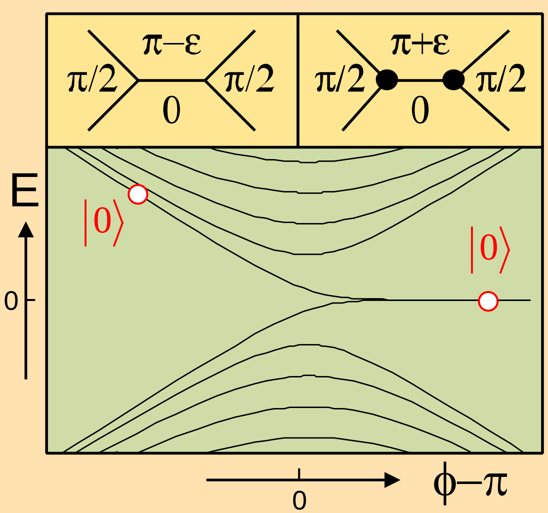
A storage register for Majoranas





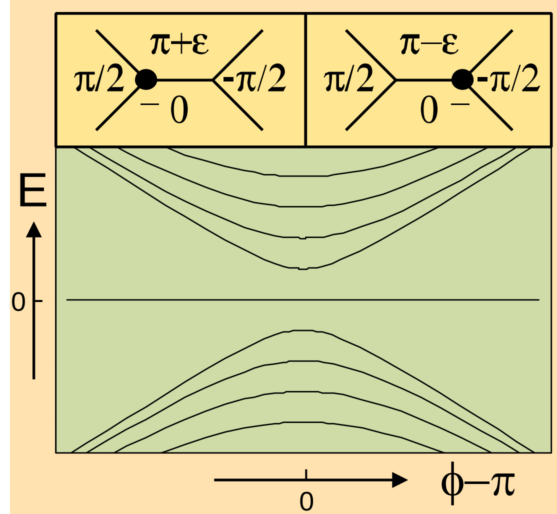
Create

A pair of Majorana bound states can be created from the vacuum in a well defined state |0>.



Braid

A single Majorana can be moved between junctions. Allows braiding of multiple Majoranas

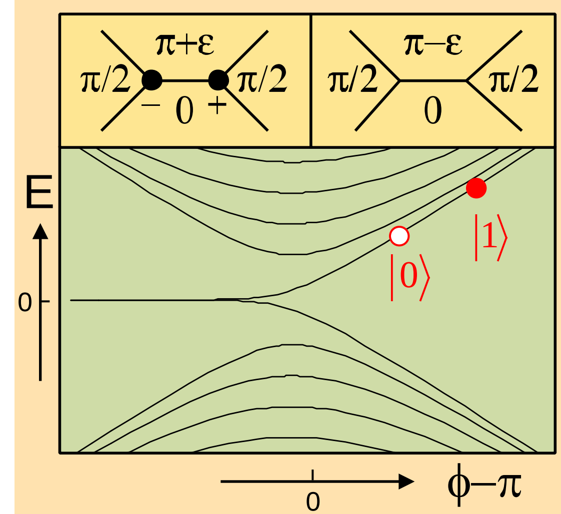


Measure

Fuse a pair of Majoranas.

States |0,1> distinguished by

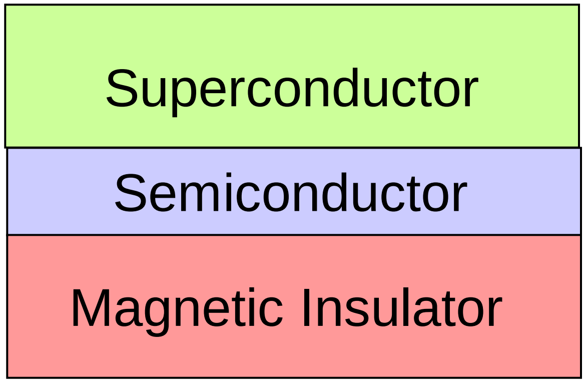
- presence of quasiparticle.
- supercurrent across line junction

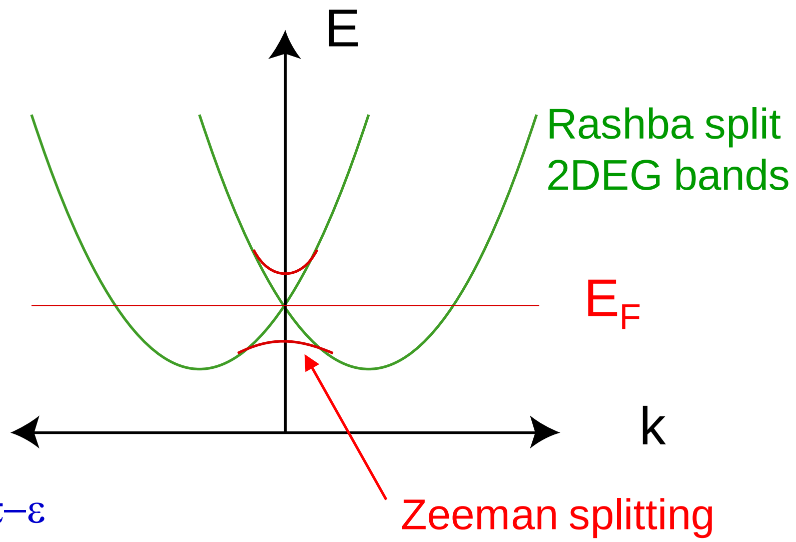


Another route to the 2D p+ip superconductor

Semiconductor - Magnet - Superconductor structure

Sau, Lutchyn, Tewari, Das Sarma ‘09





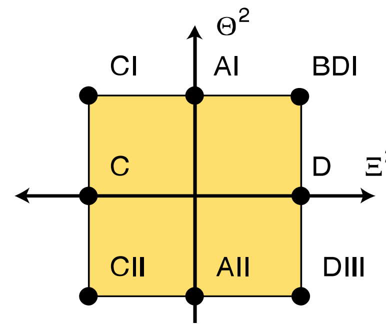
- Single Fermi circle with Berry phase $\pi-\varepsilon$
- Topological superconductor with Majorana edge states and Majorana bound states at vortices.
- Variants :
  - use applied magnetic field to lift Kramers degeneracy (Alicea ’10)
  - Use 1D quantum wire (eg InAs). A route to 1D p wave superconductor with Majorana end states. (Oreg, von Oppen, Alicea, Fisher ’10
- Challenge : requres very low electron density → high purity.

Periodic Table of Topological Insulators and Superconductors

Anti-Unitary Symmetries :

- Time Reversal : $\Theta H(\mathbf{k})\Theta^{-1} = +H(-\mathbf{k})$; $\Theta^2 = \pm 1$
- Particle - Hole : $\Xi H(\mathbf{k})\Xi^{-1} = -H(-\mathbf{k})$; $\Xi^2 = \pm 1$

Unitary (chiral) symmetry : $\Pi H(\mathbf{k})\Pi^{-1} = -H(\mathbf{k})$; $\Pi \propto \Theta\Xi$



8 antiunitary symmetry classes

Symmetry				d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Complex K-theory (A, AIII)

Real K-theory (AI – CI)

Altland-Zirnbauer Random Matrix Classes

Bott Periodicity d→d+8

Kitaev, 2008
Schnyder, Ryu, Furusaki, Ludwig 2008