On the spatial dependence of extreme ocean storm seas

Slides at *www.lancs.ac.uk/∼jonathan*

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Acknowledgement

- Shell Statistics and Data Science, Shell metocean group
- Department of Mathematics and Statistics, Lancaster University

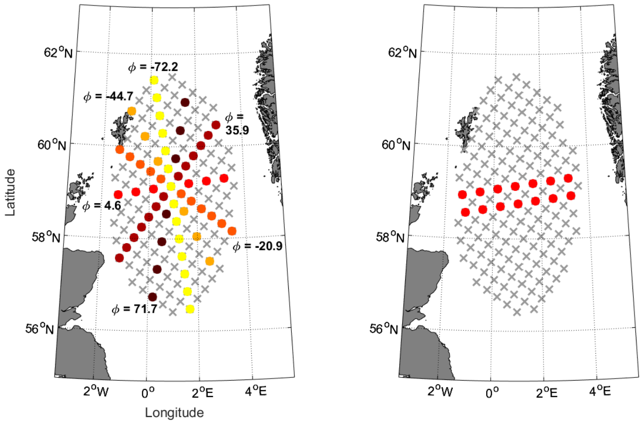
- Pre-print (Ross et al. 2017a, accepted for Ocean Engineering) with full references and acknowledgements

Context

- Rational and consistent design and assessment of marine structures
  - Reduce bias and uncertainty in estimation of structural integrity
  - Quantify uncertainty as well as possible
- Non-stationary marginal, conditional and spatial extremes
  - Multiple locations, multiple variables, time-series
  - Multidimensional covariates
- Rational quantification of uncertainty
  - Data acquisition (simulator or measurement)
  - Data pre-processing (storm peak identification)
  - Threshold uncertainty
  - Model form (marginal measurement scale effect, spatial extremal dependence)
- Improved understanding and communication of risk
  - Incorporation within established engineering design practices
  - Knock-on effects of improved inference

Motivation : North Sea application

Storm peak H_S from gridded NEXTRA *winter* storm hindcast for North Sea locations; directional variability in storm severity; "strips" of locations with different orientations; central location for directional model



Modelling extremal spatial dependence : why bother?

- Improved inference for the characteristics of extremes at one location exploiting data from multiple locations in a spatial neighbourhood
- Improved estimation of risk for spatially-distributed structures (coastal defences, multiple installations) from spatially spread storm events

- Can we estimate spatial extremes models usefully from typical metocean hindcast data?
- Can we see evidence for covariate effects in extremal spatial dependence for ocean storm severity?

Modelling extremal spatial dependence : mathematically

- Locations $j = 1, 2, ..., p$, continuous random variables and values $\{X_j\}$, $\{x_j\}$
- Spatial distribution of storm peak H_S

$$f(x_1, x_2, ..., x_p) = [f(x_1)f(x_2)...f(x_p)]\, \mathcal{C}(x_1, x_2, ..., x_p)$$

- $\{f(x_j)\}$ are marginal densities, $\mathcal{C}(x_1, x_2, ..., x_p)$ is dependence "copula"
- Interested in estimating things like "the shape of an extreme storm"

$$f(x_1, x_2, ..., x_p | X_k = x_k > u_k) \text{ for large } u_k$$

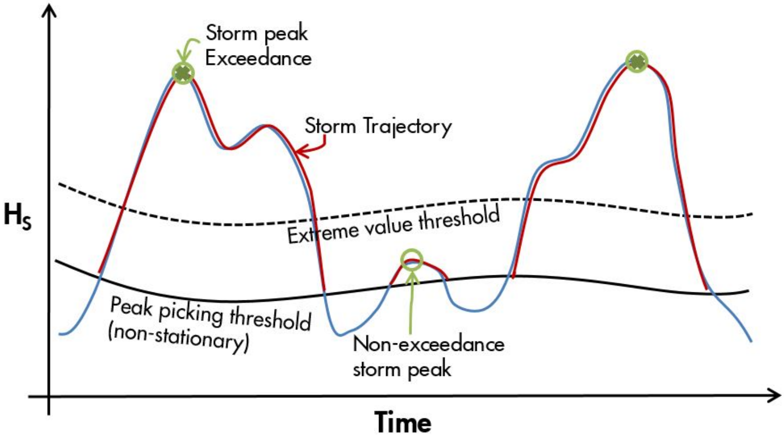
- We know how to estimate extremes marginally, but what about extremal dependence?
- $\Rightarrow$ study spatial extremes, i.e. sensible models for $\mathcal{C}(x_1, x_2, ..., x_p)$

Modelling extremal spatial dependence : procedure

- Sample of peaks over threshold $\{y\}$ at p locations, with covariates $\{\theta\}$
- Simple marginal gamma-GP model
- Sample transformed ("whitened") to standard Frechet scale per location
- Spatial extremes ("max-stable model") to estimate extremal spatial dependence
- Bayesian inference estimating joint distributions of parameters, uncertainties

Marginal : isolating storm peaks

$H_S \approx 4\times$ standard deviation of ocean surface time-series at specific location corresponding to a specified period (typically three hours)



Marginal : gamma-generalised Pareto

- Simple marginal gamma-GP model fitted using Bayesian inference
- GP ξ, σ, gamma α, β, and threshold ψ all functions of θ
- Spline parameterisation for model parameters in terms of θ
- ψ for pre-specified threshold probability τ

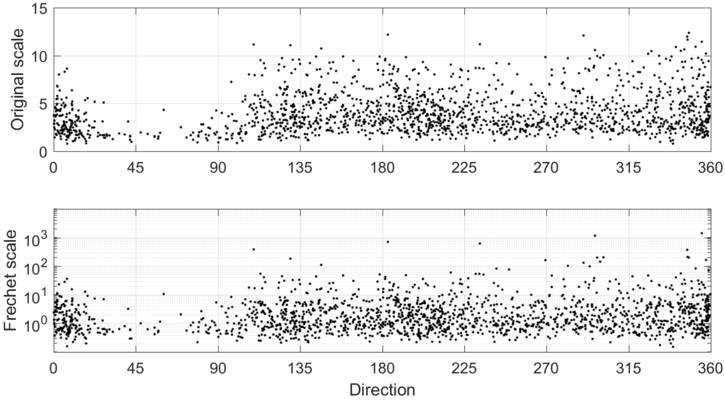
- Gibbs sampling when full conditionals available
- Otherwise Metropolis-Hastings (MH) within Gibbs, using suitable proposal mechanisms

- Sample of joint posterior of $\{\xi_\theta, \sigma_\theta, \alpha_\theta, \beta_\theta, \psi_\theta\}$ estimated

- **Ross et al. [2017b]**, Frigessi et al. [2002], Behrens et al. [2004], MacDonald et al. [2011]

Marginal : transformation to standard Fréchet scale

Storm peak H_S on direction for central location before and after standardisation to Fréchet scale



Extremes basics : marginal

- Block maxima Y_k at location k have distribution F_{Y_k} which is "max-stable" in the sense that $F^n_{Y_k}(b'_{kn} + a'_{kn} y_k) = F_{Y_k}(y_k)$ for some sequences $\{a'_{kn} > 0\}$ and $\{b'_{kn}\}$
- Only limiting distribution for F_{Y_k} is generalised extreme value (GEV)

$$\begin{aligned} F_{Y_k}(y_k) &= \exp[-\exp\{(y_k - \eta)/\tau\}] \text{ for } \xi = 0 \\ &= \exp[-\{1 + \xi(y_k - \eta)/\tau\}_+^{-1/\xi}] \text{ otherwise} \end{aligned}$$

Extremes basics : spatial

- Similarly, F_Y for block maxima Y at p locations "max-stable" when $F_Y^n(b'_{1n} + a'_{1n}y_1, b'_{2n} + a'_{2n}y_2, ..., b'_{pn} + a'_{pn}y_p) = F_Y(y_1, y_2, ..., y_p)$
- Transform to unit Fréchet $Z_k = \{1 + \xi(Y_k - \eta)/\tau\}^{1/\xi}$, $F_{Z_k}(z_k) = \exp(-1/z_k)$, for $z_k > 0$. Then

$$F_Z(z_1, z_2, ..., z_p) = F_Z(nz_1, nz_2, ..., nz_p)^n$$

- Only choices of F_Z exhibiting this "homogeneity" correspond to finite-dimensional distributions from max-stable processes (MSPs), and are hence valid for spatial extreme value modelling

Spatial : basic theory

- Max-stable process (MSP): a means of extending the GEV for modelling maxima at one location, to multivariate extreme value distributions for modelling of component-wise maxima observed on a lattice
- On unit Fréchet scale, only choices of F_Z exhibiting "homogeneity" are valid for spatial extreme value modelling
- Convenience: "exponent measure" V_Z

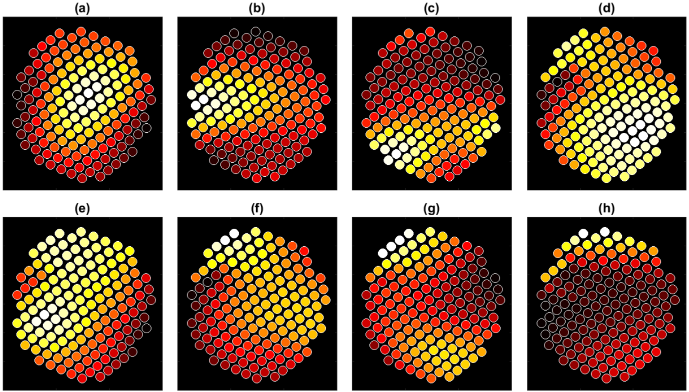
$$F_Z(z_1, z_2, ..., z_p) = \exp\{-V_Z(z_1, z_2, ..., z_p)\}$$

- Convenience: "extremal coefficient" θ_p

$$\begin{aligned} F_Z(z, z, ..., z) &= \exp\left(-V_Z(z, z, ..., z)\right) \\ &= \exp\left(-z^{-1} V_Z(1, 1, ..., 1)\right) \text{ from the homogeneity property} \\ &= \exp\left(-\theta_p / z\right) \end{aligned}$$

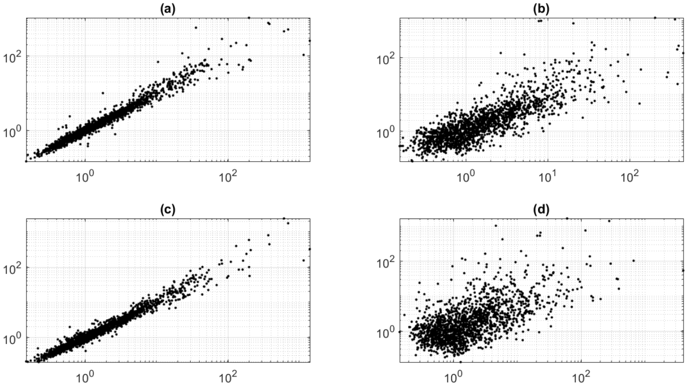
Spatial : data

Fréchet scale observations of the spatial distribution of storm peak H_S over the North Sea spatial grid for 8 typical events (a)-(h). The spatial maximum for each event is given as a white disc, and the spatial minimum as a black disc (with white outline). The white $\rightarrow$ yellow $\rightarrow$ red $\rightarrow$ black colour scheme indicates the spatial variation of relative magnitude of storm peak H_S



Spatial : data

Fréchet scale scatter plots of storm peak H_S for different pairs of locations. Panel (a) for the central location and its nearest neighbour to the West along the approximate West-East transect with angle $\phi = 4.6$; panel (b) for the end locations of the same transect. Panel (c) for the central location and its nearest neighbour to the North along the approximate North-South transect with angle $\phi = -72.2$; panel (d) for the end locations of the same transect



Spatial : V_Z for Smith, Schlather and Brown-Resnick processes

- **Smith** : For two locations s_k, s_l in $\mathcal{S}$, V_{kl} for Smith process given by

$$V_{kl}(z_k, z_l; h(\Sigma)) = \frac{1}{z_k}\Phi(\frac{m(h)}{2} + \frac{\log(z_l/z_k)}{m(h)}) + \frac{1}{z_l}\Phi(\frac{m(h)}{2} + \frac{\log(z_k/z_l)}{m(h)})$$

- $h = s_l - s_k$, $m(h)$ is Mahalanobis distance $(h'\Sigma^{-1}h)^{1/2}$ between s_k and s_l
- Σ is 2×2 covariance matrix (2-D space) to be estimated. Σ scalar in 1-D
- $V_{kl}(1, 1; h(\Sigma)) = 2\Phi(m(h)/2)$ by construction

- **Schlather** similar likelihood, parameterised in terms of Σ only
- **Brown-Resnick** identical likelihood, parameterised in terms of Σ and scalar Hurst parameter H (estimated up front)

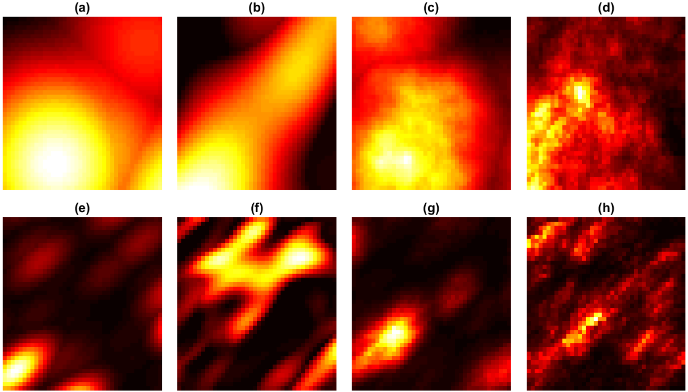
Spatial : Constructive representation

- MSP is maximum of multiple copies $\{W_i\}$ $(i \geq 1)$ of random function W
- Each W_i weighted using Poisson process $\{\rho_i\}$ $(i \geq 1)$.
- The MSP $Z(s)$ for s in spatial domain $\mathcal{S}$ is $Z(s) = \mu^{-1} \max_i\{W_i^+(s)/\rho_i\}$
- $W_i^+ = \max\{W_i(s), 0\}$
- $\mu = E(W^+(s)) = 1$ by construction typically
- $\rho_i = \epsilon_i$ for $(i = 1)$, $\rho_i = \epsilon_i + \rho_{i-1}$ for $(i > 1)$, and $\epsilon_i \sim \text{Exp}(1)$
- Different choices of $W(s)$ give different MSPs.

- **Smith** : $W_i(s; s_i, \Sigma) = \varphi(s - s_i; \Sigma)/f_S(s_i)$, with s_i sampled from density $f_S(s_i)$ on $\mathcal{S}$, with φ representing standard Gaussian density
- **Schlather, Brown-Resnick** : Similar

Spatial : illustrations

Illustrative realisations of Smith (a,e), Schlather (b,f), and Brown-Resnick (c,d,g,h) processes for different parameter choices. The first row corresponds to parameter settings $(\Sigma_{11}, \Sigma_{22}, \Sigma_{12}) = (300, 300, 0)$ for all processes, and the second row to (30,20,15). For Brown-Resnick processes (c,g), Hurst parameter $H = 0.95$. For Brown-Resnick processes (d,h), $H = 0.65$. Each panel can be considered to show a possible spatial realisation of storm peak H_S, similar to those shown earlier



Spatial : estimation approximations

- Theory gives us models for pairs of locations. Cannot write down full joint likelihood $\ell(\Sigma; \{y_j\})$. Approximate with "composite" likelihood $\ell_C(\Sigma; \{y_j\})$

$$\ell(\Sigma; \{y_j\}) \approx \ell_C(\Sigma; \{y_j\}) = \sum_{\{k,l\} \in \mathcal{N}} w_{kl} \log f_{kl}(y_k, y_l; h(\Sigma))$$

- Theory applies for block maxima Z, but we have peaks over threshold Y. For $y_k, y_l > u$ for large u, approximate

$$\Pr[Y_k \leq y_k, Y_l] \approx \Pr[Z_k \leq y_k, Z_l]$$

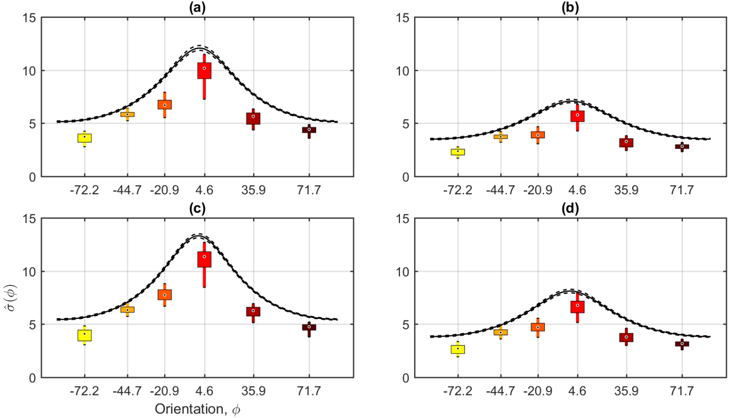
- Need $f_{kl}(y_k, y_l; h(\Sigma))$ for non-exceedances of u also, so make "censored" likelihood approximation

Spatial : estimation

- Estimate joint distribution of $\Omega = [\Sigma_{11}, \Sigma_{22}, \Sigma_{12}]$ (2-D space, or $\Omega = \Sigma$ in 1-D)
- MCMC using Metropolis-Hastings
  - Current state Ω_{r-1}, marginal posterior $f_M(\beta_M)$, original sample D of storm peak H_S.
  - Draw a set of marginal parameters β_{Mr} from f_M, independently per location.
  - Use β_{Mr} to transform D to standard Fréchet scale, independently per location, obtaining sample D_{Fr}.
  - Execute "adaptive" MCMC step from state Σ_{r-1} with sample D_{Fr} as input, obtain Σ_r.
- Adaptive MCMC candidates generated using $\Omega_r^c = \Omega_{r-1} + \gamma\epsilon_1 + (1-\gamma)\epsilon_2$
  - $\gamma \in [0,1]$, $\epsilon_1 \sim N(0, \delta_1^2 I_3/3)$, $\epsilon_2 \sim N(0, \delta_2^2 S_{\Omega_{r-1}}/3)$
  - $S_{\Omega_{r-1}}$ estimate of variance of Ω_{r-1} using samples to trajectory to date
  - Roberts and Rosenthal [2009]

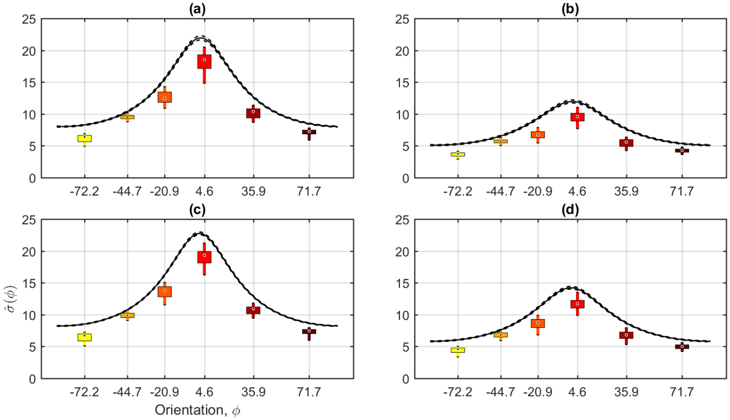
Spatial : $\hat{\sigma}(\phi)$ for Smith

For all transects with a given orientation ϕ estimated using 1-D (box-whisker) and 2-D (black) Smith processes. ϕ is quantified as the transect angle anticlockwise from a line of constant latitude. The first (second) row: marginal threshold non-exceedance probability 0.5 (0.8). The first (second) column: censoring threshold non-exceedance probability 0.5 (0.8). For 1-D estimates with a given ϕ, box centres = median, box edges = 0.25 and 0.75 quantiles across all parallel transects; whisker edges = 0.025 and 0.975 quantiles. For 2-D estimates, the 0.025, 0.5 and 0.975 quantiles are shown as a function of ϕ. Note that the colour coding of box-whisker plots corresponds to that of transect orientation



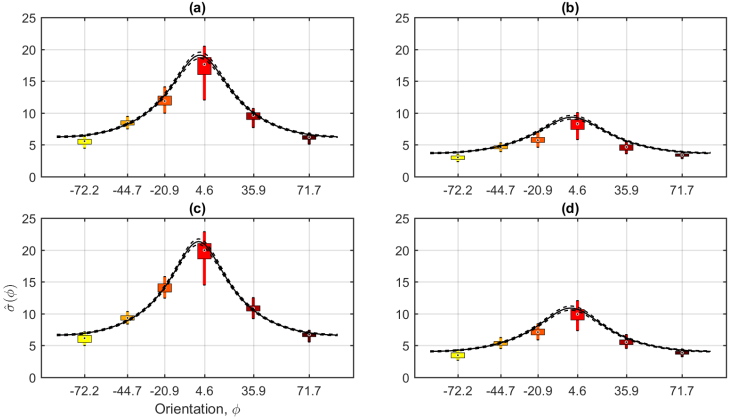
Spatial : $\hat{\sigma}(\phi)$ for Schlather

Estimated extremal spatial dependence parameter $\hat{\sigma}(\phi)$ for all transects with a given orientation ϕ estimated using 1-D (box-whisker) and 2-D (black) Schlather processes. ϕ is quantified as the transect angle anticlockwise from a line of constant latitude. The first (second) row corresponds to a choice of marginal threshold with non-exceedance probability 0.5 (0.8). The first (second) column corresponds to a choice of censoring threshold with non-exceedance probability 0.5 (0.8)



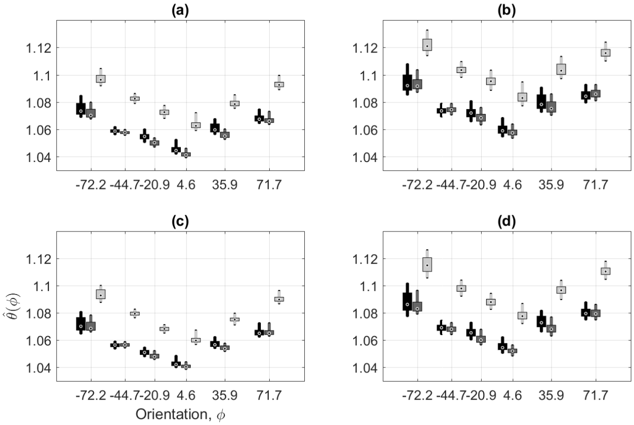
Spatial : $\hat{\sigma}(\phi)$ for Brown-Resnick

Estimated extremal spatial dependence parameter $\hat{\sigma}(\phi)$ for all transects with a given orientation ϕ estimated using 1-D (box-whisker) and 2-D (black) Brown-Resnick processes with $H = 0.75$. ϕ is quantified as the transect angle anticlockwise from a line of constant latitude. The first (second) row corresponds to a choice of marginal threshold with non-exceedance probability 0.5 (0.8). The first (second) column corresponds to a choice of censoring threshold with non-exceedance probability 0.5 (0.8)



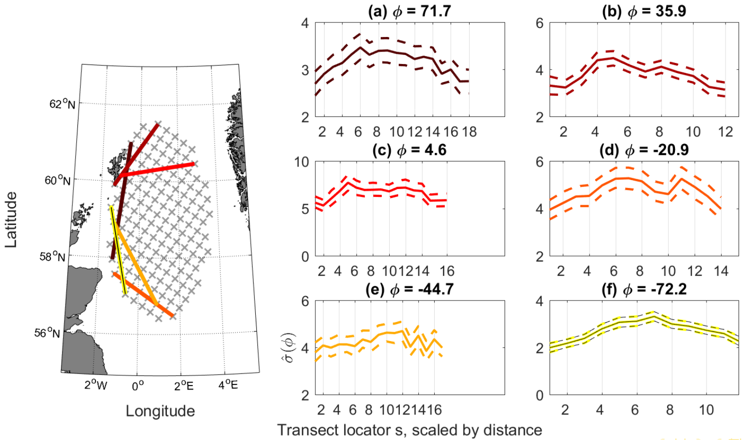
Spatial : extremal coefficient $\hat{\theta}(\phi)$

Estimated extremal coefficient $\hat{\theta}(\phi)$ for all transects with a given orientation ϕ, estimated using 1-D Smith (black), Schlather (dark grey) and Brown-Resnick (light grey) processes. The first (second) row corresponds = marginal threshold with non-exceedance probability 0.5 (0.8). The first (second) column = censoring threshold with non-exceedance probability 0.5 (0.8)



Spatial : spatial dependence parameter $\hat{\sigma}(\phi, s)$ for individual transects

Smith process with marginal and censoring thresholds = non-exceedance probability of 0.8. (b)-(g): $\hat{\sigma}(\phi, s)$ for fixed orientation ϕ (given in the panel title) as a function of transect locator s. (a): transects with $s = 1$ for different orientations ϕ. (b)-(g): abscissa values for transect locators are scaled to physical perpendicular distances between parallel transects



Discussion

- Possible to estimate reasonable spatial extremes models for typical samples of hindcast data
- Consistent inferences from Smith, Schlather and Brown-Resnick models
- Evidence for directional and spatial anisotropy

- Only investigated "asymptotically dependent" models here, but see Keresztur et al. [2016]
- Did not perform simultaneous marginal and dependence inference
- Essential that marginal modelling performed thoughtfully

- Fetch effects may be visible
- Other locations, basins

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