# Inference for directional ocean wave models

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## What does a stochastic model for ocean waves look like?

We want to model the displacement of the surface of the ocean over space and time. The frequency-direction spectrum, $S(\omega, \phi)$, is the typical quantity of interest for second-order statistics of ocean waves. The frequency-direction spectrum can be written as

$$S(\omega, \phi) = f(\omega)D(\omega, \phi)$$

where $f(\omega)$ is the marginal spectral density function and $D(\omega, \phi)$ is the spreading function. For example:

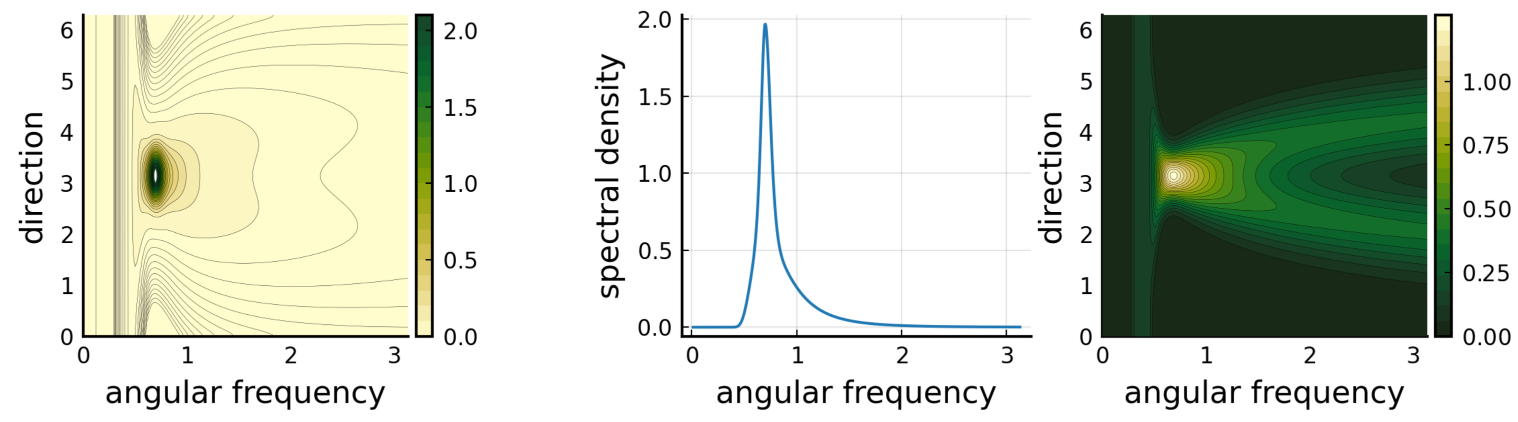



Fig. 1: Example $S(\omega, \phi)$ (left) decomposed into $f(\omega)$ (centre) and $D(\omega, \phi)$ (right)

For wind-sea waves, these components can be modelled by

$$f(\omega) = \alpha\omega^{-r} \exp\left\{-\frac{r}{4}\left(\frac{|\omega|}{\omega_p}\right)^{-4}\right\}\gamma^{\delta(\omega)}$$

and

$$D(\omega, \phi) = \frac{1}{2\sigma(\omega)\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \sum_{i=1}^{2} \exp\left\{-\frac{1}{2}\left(\frac{\phi - \phi_{mi}(\omega) - 2\pi k}{\sigma(\omega)}\right)^2\right\}$$

where

$$\phi_{m1}(\omega) = \phi_m + \beta \exp\{-\nu \cdot \min(\omega_p/|\omega|, 1)\}/2$$
$$\phi_{m2}(\omega) = \phi_m - \beta \exp\{-\nu \cdot \min(\omega_p/|\omega|, 1)\}/2$$
$$\sigma(\omega) = \sigma_l - \sigma_r\left[4\left(\omega_p/|\omega|\right)^2 - \left(\omega_p/|\omega|\right)^8\right]/3$$

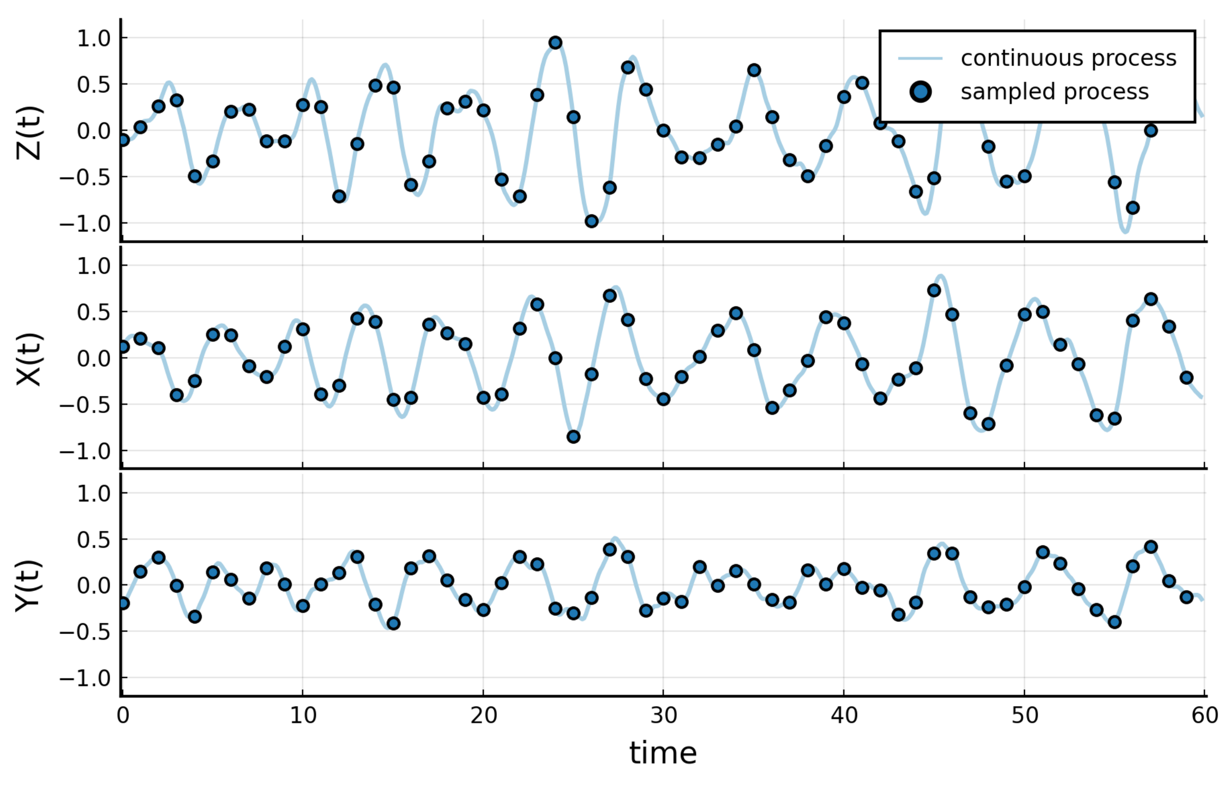
As a result, we have

- location parameters: $\omega_p, \phi_m$
- shape parameters: $\alpha, \gamma, r, \beta, \nu, \sigma_l, \sigma_r$.

Parameter estimation in this setting is challenging, as there are many parameters, and they are non-stationary over time.

## What kind of data do we get?

It is usually not practical to record the full random field. Instead, we can record time series of the 3D displacement of a buoy. Though this process evolves continuously in time, we can only record it discretely. An example of such a series is shown below:




Such data can be described by the spectral density matrix function:

$$\boldsymbol{f}(\omega) = \begin{bmatrix} f_{zz}(\omega) & f_{zx}(\omega) & f_{zy}(\omega) \\ f_{xz}(\omega) & f_{xx}(\omega) & f_{xy}(\omega) \\ f_{yz}(\omega) & f_{yx}(\omega) & f_{yy}(\omega) \end{bmatrix} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boldsymbol{c}(\tau)e^{-i\omega\tau}\mathrm{d}\tau,$$

where $\boldsymbol{c}(\tau)$ is the autocovariance of the 3D displacement process.

## How does this displacement data relate to the model for the surface?

The frequency-direction spectrum and spectral density matrix function are related by a transfer function $G(\omega, \phi) = \left[1 \;\; i\cos\phi \;\; i\sin\phi\right]^T$ so

$$\boldsymbol{f}(\omega) = \int_0^{2\pi} G(\omega, \phi)G(\omega, \phi)^H S(\omega, \phi)\mathrm{d}\phi.$$

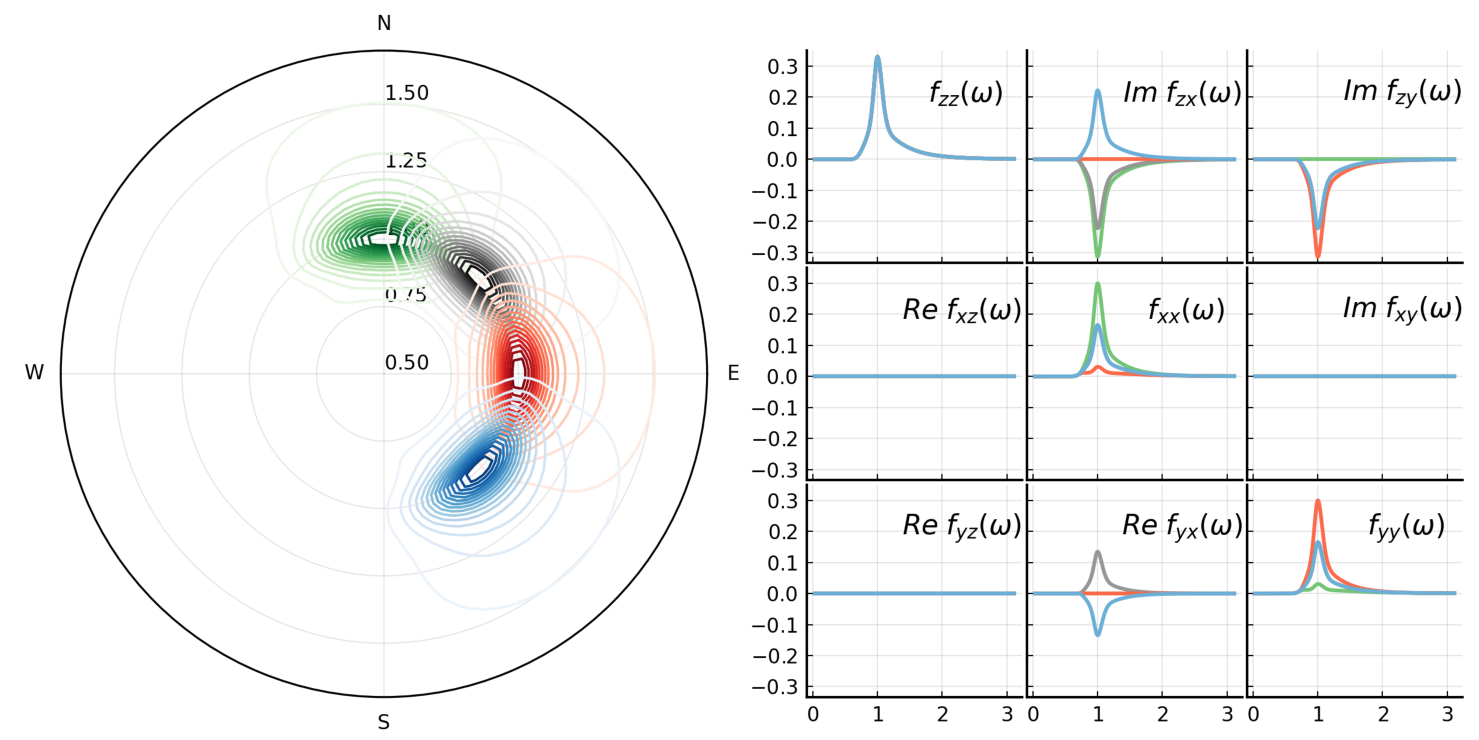



Fig. 3: Example of the relation between $S(\omega, \phi)$ (left) and $\boldsymbol{f}(\omega)$ (right).

## What about inference?

Current techniques use non-parametric estimates of the frequency-direction spectrum alongside least squares curve fitting to estimate model parameters. In contrast, we propose a novel likelihood-based method which fits directly to the data, bypassing the need to estimate $S(\omega, \phi)$ non-parametrically.

**Current technique**

3D data → Estimate $S(\omega, \phi)$ non-parametrically → Fit model to the estimate of $S(\omega, \phi)$ using least squares

**New technique**

Model for $S(\omega, \phi)$ → Convert to model for $\boldsymbol{f}(\omega)$ using transfer function → Fit model for $\boldsymbol{f}(\omega)$ directly to 3D data using the debiased Whittle likelihood

## Comparison via simulation studies

The table below shows the results of a simulation study comparing our technique against the best of the current techniques, showing percentage relative bias and standard deviation (std). Parameters are grouped into location and shape and the bias and std are averaged to improve clarity.

| parameter | current | new |
|---|---|---|
| location bias | 0.02% | 0.01% |
| location std | 1.41% | 0.68% |
| shape bias | 28.61% | 0.51% |
| shape std | 30.25% | 7.70% |

- Improvement is minor for location parameters, but major for shape.
- Our novel method is statistically more powerful and resolves more parameters thus providing a better characterisation of the ocean.
- Such results can be further used for better forecasting and decision making in ocean engineering and environmental monitoring.

**References:**

Grainger, J. P., Sykulski, A. M., Jonathan, P., & Ewans, K. (2021). Estimating the parameters of ocean wave spectra. Ocean Engineering, 229, 108934.

Grainger, J. P., Sykulski, A. M., Ewans, K., Hansen, H. F., & Jonathan, P. (2022). A multivariate pseudo-likelihood approach to estimating directional ocean wave models. arXiv preprint arXiv:2202.03773.

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