Subsonic and Supersonic Flow Through Pitot Tubes

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Abstract

In the following work, the differences between calculations of velocity in subsonic and supersonic flows are examined. First, the equations relating to subsonic, incompressible flow are derived and applied to flow through a Pitot tube to derive an expression for the velocity. Secondly, equivalent equations relating to subsonic, compressible flow are derived and a discussion of the error generated by using an equation derived for incompressible flow to calculate velocity in a compressible flow follows. The final section considers supersonic flow and the effects of shock waves on the calculation of velocity from a Pitot tube.
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1 Introduction

First invented by Henri Pitot in the 18th century, a Pitot tube (or Pitot-Static tube) is a device used to measure pressure, from which, the speed of the fluid of interest can be calculated. Perhaps the most common use, and the one considered here, is the determination of airspeed in aircraft. There are also industrial applications for measuring flow velocities of various fluids using Pitot tubes, including measuring high speed wind at a weather station or determining the volume of air used by an air conditioning unit.

The airspeed calculated by the Pitot tube is used by the pilot as a reference point for take-off and landing speeds, and to make adjustments to the way the aircraft is flying. Given that the calculated airspeed is given directly to the pilot through the airspeed indicator [1], minimising the error in calculation of airspeed is vitally important for safe flight. A large difference between the calculated airspeed and true airspeed could lead to incorrect reactions from flight crew. Errors in the calculation can be generated in two ways: a physical problem with the Pitot Tube such as a blockage in the tube, possibly created by insects or icing; alternatively, errors may be generated from assumptions made during the calculations. This project seeks to identify where potentially poor assumptions may be made and what their impact on the calculated value of speed may be.

The reason Pitot tubes are still used instead of GPS is because Pitot tubes measure airspeed rather than ground speed (which is measured by GPS). GPS calculates how fast the aircraft is flying relative to the ground whereas Pitot tubes calculate airspeed which is how fast the aircraft is flying relative to the surrounding air i.e it takes into account wind. Furthermore, any attempt to use a GPS reading and take into account the altitude of the aircraft would also rely upon the use of an altimeter. This reliance of one device upon another can increase the risk of error in calculation.

1.1 Aims and Objectives

The objectives of this project are to investigate and understand the equations which define subsonic and supersonic flow through Pitot tubes (in one-dimension) and describe the assumptions made (and their validity) about the fluid in deriving such equations. In order to achieve these objectives, the aims of this project are as follows:

- To derive an equation involving pressure which explains the velocity of a steady, inviscid, incompressible fluid (and prove the validity of any assumptions made) at subsonic speeds
- To derive an equation involving pressure which explains the velocity of a steady, inviscid, compressible fluid at subsonic speeds
- To investigate the effects of the assumption of incompressibility in the example of air-flow through a Pitot Tube
- To consider the different types of shock waves found in supersonic flow and the effects they
have on the fluid
• To derive a set of equations which define the properties of the flow before and after a Normal shock wave and consider how this relates to flow through a Pitot tube
• To derive the Rayleigh Supersonic Pitot Equation and discuss its validity

1.2 Assumed Knowledge of Fluid Dynamics

Throughout the following project, the assumption that the fluid is inviscid remains; and only the validity of the assumption of whether the fluid can be approximated as incompressible is investigated (for subsonic flow). Additionally, although the aircraft is a moving object, in this work the frame of reference is transformed such that the aircraft is considered stationary and the fluid is moving around it.

1.2.1 Definitions

Knowledge of the following definitions is assumed throughout:

**Compressible Fluid**: A fluid in which the density may vary.

**Enthalpy**: A measurement of the energy in a system given by the sum of the internal energy of the system and the ratio of pressure to density. (The ‘internal energy of a gas is the energy stored in it by virtue of its molecular motion’ [2] (page 7).)

**Entropy**: A measure of the amount of energy which is unavailable to do work in a system.

**Hypersonic Flow**: A flow in which the Mach number is very large. This is usually taken to be $M > 5$.

**Incompressible Fluid**: A fluid in which the density of each particle is constant. This is equivalent to saying the flow is non-divergent i.e. $\nabla \cdot \mathbf{u} = 0$.

**Inviscid Fluid**: A fluid in which the viscous forces are zero.

**Mach Number** ($M$): the ratio of a fluid’s speed, $u$, to the local speed of sound, $c$

$$M = \frac{|u|}{c}$$

where the value of the local speed of sound is given by:

$$c = \sqrt{\gamma RT}$$

where $T$ is the absolute temperature, $R$ is the gas constant, $\gamma$ is the ratio of specific heats (approximately 1.4 for air).

**Shock Wave**: When a fluid travels faster than the local speed of sound, the sections of sound waves with high amplitude start to propagate at a higher speed than the sections of the sound
wave with lower amplitude. When the fluid is a gas (e.g. air), this leads to a boundary over which there is large, sudden change in pressure. This discontinuity boundary is the shock wave.

**Stagnation Point:** A point in a flow at which \( \mathbf{u} = 0 \).

**Steady Flow:** A flow in which the velocity \( \mathbf{u} \) does not depend on time. This is equivalent to saying \( \partial \mathbf{u} / \partial t = 0 \). (Note that this does not necessarily mean the particles are not accelerating. Since, by using the definition of the material derivative given below in Section 1.2.2, if \( \partial \mathbf{u} / \partial t = 0 \) then \( D\mathbf{u} / Dt = \mathbf{u} \cdot \nabla \mathbf{u} \), which may be non-zero.)

**Streamline:** Streamlines are curves which are tangent to the velocity field \( \mathbf{u} \).

**Subsonic Flow:** A flow in which the Mach number is less than 1.

**Supersonic Flow:** A flow in which the Mach number is greater than 1.

**Transonic Flow:** A flow in which some of the flow around an object is subsonic and in other areas is supersonic.

### 1.2.2 Equations

Additionally, knowledge of the following equation is also assumed throughout:

**Material Derivative** (of a property \( F \)):

\[
\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \mathbf{u} \cdot \nabla F
\]

Note if \( \frac{DF}{Dt} = 0 \) then property \( F \) is materially conserved.

It should be noted that in a control volume (in particular, volume \( A \) in Section 3.1.1) the properties momentum, mass and energy are materially conserved. In particular, the momentum entering the system is equal to the momentum leaving the system. Likewise with mass and energy.
2 Subsonic Flow

2.1 Derivation of Euler’s Equation

Before deriving Bernoulli’s Theorem, Euler’s equation must first be derived. Assume the fluid is steady, inviscid and homogeneous (a fluid in which the density, \( \rho \), is constant). Defining momentum, \( I \), as follows:

\[
I = \int_V \rho u \, dV
\]

where \( V \) is some fixed, arbitrary volume, and \( m \) is the mass of the fluid in the volume \( V \).

Then taking the material derivative [3] (pages 24-25):

\[
\frac{DI}{Dt} = \frac{D}{Dt} \int_V \rho u \, dV
\]

\[
\frac{DI}{Dt} = \int_V \rho \frac{Du}{Dt} \, dV
\]

where \( \frac{DI}{Dt} \) is the material derivative of momentum [3] (page 11).

By considering the fact that this equivalent to the total force acting on the fluid which comes from a normal pressure on the fluid boundary, \( S \), and an external force \( F \), assuming the fluid is ideal i.e. there are no tangential components to the force on the boundary [3] (page 25).

\[
\frac{DI}{Dt} = \int_V \rho F \, dV - \int_S \rho n \, dS
\]

with \( n \) as the unit vector, normal to the surface, \( S \). By Gauss’ Theorem:

\[
\frac{DI}{Dt} = \int_V \rho F \, dV - \int_V \nabla p \, dV
\]

\[
\int_V \rho \frac{Du}{Dt} \, dV = \int_V \rho F \, dV - \int_V \nabla p \, dV
\]

\[
\int_V \left( \rho \frac{Du}{Dt} + \nabla p - \rho F \right) \, dV = 0
\]

Since \( V \) is arbitrary, this becomes:

\[
\rho \frac{Du}{Dt} + \nabla p - \rho F = 0
\]
Rearranging and substituting in the definition of $\frac{Du}{Dt}$:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho} \nabla p + F$$  \hspace{1cm} (1)

Equation (1) is Euler’s equation.

### 2.2 Derivation of Bernoulli’s Theorem for Steady, Incompressible Flow

If it is taken that the only force acting on the object is gravity, such that $F = g = (0, 0, -g)$. Therefore it can be written that $[3]$ (page 25):

$$F = -\nabla g$$

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho} \nabla p + g$$

The assumption that the flow is steady reduces this to:

$$u \cdot \nabla u = -\frac{1}{\rho} \nabla p + g$$

Using the following common vector identity:

$$\frac{1}{2} \nabla |u|^2 = u \times (\nabla \times u) + u \cdot \nabla u$$

The expression becomes:

$$\frac{1}{2} \nabla |u|^2 - u \times (\nabla \times u) = -\frac{1}{\rho} \nabla p + g$$

Rearranging:

$$u \times (\nabla \times u) = \frac{1}{2} \nabla |u|^2 + \frac{1}{\rho} \nabla p - g$$

For homogeneous, incompressible flow (since density is constant with respect to pressure) and without loss of generality taking the constant to be equal to 0 (as it disappears by taking $\nabla$) this simplifies to:

$$u \times (\nabla \times u) = \nabla \left\{ \frac{1}{2} |u|^2 + \frac{p}{\rho} + gz \right\}$$

$$u \times (\nabla \times u) = \nabla H$$

where $H = \frac{1}{2} |u|^2 + \frac{p}{\rho} + gz$.  

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Applying the dot product to the above gives:

\[ \mathbf{u} \cdot (\mathbf{u} \times (\nabla \times \mathbf{u})) = \mathbf{u} \cdot \nabla H \]

Then by orthogonality:

\[ \mathbf{u} \cdot \nabla H = 0 \]

If \( \mathbf{u} \) is denoted by \( \mathbf{u} = (u, v, w) \) then:

\[ \mathbf{u} \cdot \nabla H = u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} + w \frac{\partial H}{\partial z} = \frac{\partial H}{\partial s} = 0 \]

where \( s \) is a parameter describing the streamline. Since \( \frac{\partial H}{\partial s} = 0 \) along a streamline then \( H \) is constant along a streamline, although the value of \( H \) may differ on different streamlines \[4\] (pages 22).

**Bernoulli’s Theorem for Steady, Incompressible Flow:**

\[ H = \frac{1}{2} |\mathbf{u}|^2 + \frac{p}{\rho} + gz = \text{constant along a streamline} \quad (2) \]

### 2.3 Subsonic Incompressible Flow Through Pitot Tubes

Bernoulli’s Theorem for Steady, Incompressible Flow can be applied to the air flow through a Pitot tube, if it is assumed the flow is incompressible.

Figure 1 and Figure 2 show two different types of Pitot Tube. First, a simple Pitot Tube (which measures the stagnation pressure) along with a perforation (in the side of the aircraft) which measures the static pressure. Secondly, a Pitot-Static tube which has two intake points connected to two manometers (pressure-measuring devices)\[6\] which measure both the static and stagnation pressures.

**Point A**

Point A is a stagnation point meaning that \( \mathbf{u}_A = 0 \). The pressure, \( p \), is measurement taken at point A, say \( p_A = p_0 \). The value of \( z \) is the height of the inlet at A above the ground, say \( z_A \).

**Point B**

Point B is a point far along the streamline and, without loss of generality assuming uniform flow, \( \mathbf{u}_B = (U, 0, 0) \), where \( U > 0 \). Similarly, the pressure \( p \) is some measurement taken at the same point far along the streamline, say \( p_B = p \). The value of \( z \) is the height of B above the ground, say \( z_B \).

It is assumed the flow steady, homogeneous and inviscid, as well as incompressible. By Bernoulli’s Theorem the quantity \( H \) is constant along a streamline, in particular:
The height difference between the two intakes \((z_A - z_B)\) is so small (compared to the height of the aircraft from the ground) that it becomes negligible and so the effects of \(g\) can be ignored i.e. approximate \((z_A - z_B) = 0\).

\[
\frac{1}{2} U^2 = \frac{(p_0 - p)}{\rho}
\]

The Mach number can then be calculated as:

\[
U = \sqrt{\frac{2 (p_0 - p)}{\rho}} \tag{3}
\]
\[ M = \frac{U}{c} \]

\[ M = \sqrt{\frac{2(p_0 - p)}{\rho c^2}} \] (4)

2.3.1 Error in Assumption of \((z_A - z_B) = 0\)

In the above calculation, it is assumed that the distance between the two intake points, A and B, is small enough that it can be approximated to be zero.

The size of the average Pitot-static tube is 25cm long with a diameter of around 1cm [5]. Therefore \((z_A - z_B) \approx 0.01\text{m}\). Taking \(g\) to be \(9.81\text{ms}^{-2}\), Equation (3) becomes:

\[ U = \sqrt{\frac{2(p_0 - p)}{\rho} + 0.0981} \]

Considering the difference between the equation considering \(g\) and the one without:

\[ \sqrt{\frac{2(p_0 - p)}{\rho} + 0.0981} - \sqrt{\frac{2(p_0 - p)}{\rho}} \]

If this approximation is a valid one, then the difference should be very close to 0. The following graph shows the difference in the two equations when considering them at sea level. At sea level,
for air with constant density, $\rho$ is taken to be $1.225 \text{kgm}^{-3}$ and $p_0$ is taken to be 101325 Pa.

Figure 3: Error in Assumption of $(z_A - z_B) = 0$

Clearly, as seen in Figure 3, the error is very small, especially considering the fact most aircraft have cruising speeds in the range of $200 \text{ms}^{-1}$. Figure 3 also demonstrates that the error increases at a faster rate as the pressure at B gets closer to the static pressure. Given the extremely small difference between the two equations, the assumption that $(z_A - z_B) = 0$ is a valid one. In a simple Pitot tube with surface perforation, the distance between A and B may be slightly larger, but the error is of the same order so it is equally valid to approximate $(z_A - z_B) = 0$.

### 2.4 An Equivalent Derivation for Steady, Compressible Flow

In general, no fluid is completely incompressible but many can be approximated as incompressible, particularly at low Mach number. For an object travelling through air at higher speeds, the compressibility must be taken into account. In order to do this, the assumption that $\nabla \cdot \mathbf{u} = 0$ and $\int \frac{dp}{\rho} = \frac{p}{\rho}$ can no longer be made. Again, assuming that the difference in height between the two inlets of a Pitot tube is negligible, the case below considers only $F = 0$.

The initial assumption of an inviscid fluid combined with the neglect of the effects of heat conduction (which is valid as aerodynamic heating only becomes significant in supersonic flow), means that the flow can be considered to be adiabatic. This means that no heat is exchanged between the system and its surroundings. Given that the process is also reversible, it is said to be isentropic i.e there is no change in entropy (see Section 3.1 for further discussion on entropy).

Considering the definition of enthalpy to be a measurement of the energy in a system given by the sum of the internal energy of the system and the ratio of pressure to density; as an equation, the enthalpy of the flow $h$, can be defined in the following way [3] (page 141):
\[ h = E_l + \frac{p}{\rho} \]

where \( E_l \) is the internal energy of the flow. Ignoring gravitational effects, the stagnation enthalpy (that is, the enthalpy when the speed is zero) can be expressed as [2] (page 36):

\[
h_0 = h + \frac{1}{2} |u|^2
\]  

Equation (5) is a restatement of Bernoulli’s Theorem in terms of \( h \) rather than \( p \) and \( \rho \). An alternative definition of enthalpy is given below, where \( \gamma \) is the ratio of specific heats at constant pressure and constant volume [2] (pages 6, 37-38):

\[
h = \frac{\gamma}{\gamma - 1} RT
\]

This can also be expressed in terms of pressure due to properties of an ideal gas:

\[
h = \frac{\gamma}{\gamma - 1} \frac{p}{\rho}
\]

Similarly:

\[
h_0 = \frac{\gamma}{\gamma - 1} RT_0 = \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0}
\]

Substituting these alternative definitions into Equation (5) gives:

\[
\frac{\gamma}{\gamma - 1} \frac{p}{\rho} + \frac{|u|^2}{2} = \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0}
\]

Considering the stagnation density, for isentropic flow, the following is also true by properties of an ideal gas [2] (page 37):

\[
\frac{\rho_0}{\rho} = \left( \frac{p_0}{p} \right)^{\frac{1}{\gamma}}
\]

where \( \rho_0 \) is the stagnation density (i.e. the density at point A). In order to remove \( \rho \) from this equation (as in practice it is difficult to calculate or measure), by rearranging the equation, multiplying both sides by \( p \) and multiplying the right-hand side by \( \frac{p_0}{p_0} \), the following is obtained:

\[
\frac{p}{\rho} = \frac{p_0}{\rho_0} \left( \frac{p_0}{p} \right)^{\frac{1}{\gamma}}
\]
\[ \frac{p}{\rho} = p_0 \left( \frac{p}{p_0} \right)^{\frac{\gamma-1}{\gamma}} \]  

Substituting (7) into (6):

\[ \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0} \left( \frac{p}{p_0} \right)^{\frac{\gamma-1}{\gamma}} + \frac{|u|^2}{2} = \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0} \]  

(8) is the compressible flow equivalent to Bernoulli’s Theorem for Incompressible Flow. Rearranging and again using \( u = (U, 0, 0), U > 0 \):

\[ U = \sqrt{\frac{2\gamma}{\gamma - 1} \frac{p_0}{\rho_0} \left( 1 - \left( \frac{p}{p_0} \right)^{\frac{\gamma-1}{\gamma}} \right)} \]  

Substituting \( c^2 = \frac{\gamma p_0}{\rho_0} \):

\[ U = \sqrt{\frac{2c^2}{\gamma - 1} \left( 1 - \left( \frac{p}{p_0} \right)^{\frac{\gamma-1}{\gamma}} \right)} \]  

\[ M = \frac{U}{c} = \sqrt{\frac{2}{\gamma - 1} \left( 1 - \left( \frac{p}{p_0} \right)^{\frac{\gamma-1}{\gamma}} \right)} \]  

2.5 Comparison of Incompressible and Compressible Flow Equations

Subsonic flow can be assumed to be incompressible for a very low Mach number (in practice, often taken to be \( M < 0.3 \)). This means the object (e.g. aircraft) would be travelling at less than 30% of the local speed of sound. For reference, at room temperature and close to sea level, the speed of sound in air is approximately \( 343 \text{ms}^{-1} \). Then 30% of this speed of sound is approximately \( 103 \text{ms}^{-1} \) or 230mph. It should be noted that the maximum cruising speed of a Boeing 747-8 commercial passenger jet is 570mph (around Mach 0.85) [6]. This means that the assumption of incompressibility is not valid for most practical applications concerning airspeed.

The assumption of incompressible flow is a valid one for low \( M \) because a flow can be approximated as incompressible if the change in density of each fluid particle is very close to zero. Consider the ratio of the two pressures, \( p_0 \) and \( p \), for isentropic compressible flow by rearranging (9) [3] (page 11):

\[ \frac{p}{p_0} = \left( 1 - \frac{\gamma - 1}{2} \left( \frac{U}{c} \right)^2 \right)^{\frac{2}{\gamma - 1}} \]

\[ \frac{p}{p_0} = \left( 1 - \frac{\gamma - 1}{2} M^2 \right)^{\frac{2}{\gamma - 1}} \]
Using a binomial expansion on the right hand side gives:

\[
\frac{p}{p_0} = 1 - \frac{\gamma}{2} M^2 + \frac{\gamma}{8} M^4 - \frac{\gamma (2 - \gamma)}{48} M^6 + O \{ M^8 \}
\]

When the term of the order \(M^4\) (and consequently higher order terms) is small enough that it becomes negligible and can be approximated to zero, by substituting in for \(c^2 = \frac{p_0}{\rho_0}\), this equation can be written as:

\[
\frac{p}{p_0} + \frac{1}{2} |u|^2 = \frac{p_0}{\rho_0}
\]

This is Bernoulli’s Theorem for steady, incompressible flow, when the gravitational effects are ignored, as derived previously. Therefore, the fluid can be approximated as incompressible for low \(M\).

Rearranging the above equation, which is previously derived in Section 2.2, Equation (3), if the flow is incompressible then:

\[
\frac{p_0 - p}{\frac{1}{2} \rho U^2} = 1
\]

The quantity on the left-hand side is known as the compressibility factor. By considering \(\frac{p_0}{p}\), and recalling that:

\[
\frac{\rho_0}{\rho} = \left( \frac{p_0}{p} \right)^{\frac{1}{\gamma}}
\]

meaning:

\[
\frac{p_0}{p} = \left( \frac{\rho_0}{\rho} \right)^{\gamma}
\]

From the derivation for compressible flow in Section 2.3:

\[
\frac{\rho_0}{\rho} = \left( 1 + \frac{\gamma - 1}{2} M^2 \right)^{\frac{1}{\gamma - 1}}
\]

\[
\frac{p_0}{p} = \left( 1 + \frac{\gamma - 1}{2} M^2 \right)^{\frac{\gamma}{\gamma - 1}}
\]

\[
\frac{p_0 - p}{\frac{1}{2} \rho U^2} = \frac{\left( 1 + \frac{\gamma - 1}{2} M^2 \right)^{\frac{\gamma}{\gamma - 1}} - 1}{\frac{3}{2} M^2}
\]

The term on the left-hand-side is the compressibility factor which, if the flow is incompressible should be equal to 1. Therefore, if the flow can be approximated as incompressible, the right-hand-side should also be equal to 1. It can be shown that as \(M \to 0\) the limit of the right-hand-side is indeed 1. This means that, if the assumption of incompressibility is a valid one:
The error in the assumption of incompressibility can then be defined as the value of the left-hand-side of the above equation. Figure 4 (for the case where the gas concerned is air ($\gamma \approx 1.4$)) is a graph of the error, as $M$ increases. It can be seen that the expression is in fact only equal to 0 when $M = 0$ i.e. the fluid can only truly be approximated as being incompressible when it is stationary. The error in Figure 4 relates to the error in assuming that the compressibility factor is 1, regardless of the velocity of the fluid. The graph shows that the error increases to over 0.25 (meaning a calculation error of over 25%) which is highly significant. Clearly, as $M \to 1$, the error increases greatly and it is not valid to consider the fluid to be incompressible, particularly at high Mach number.

Figure 4: Error from Assumption of Incompressibility
3 Supersonic Flow

3.1 Shock Waves

In supersonic flow, when the speed of the fluid is faster than the local speed of sound, \( M > 1 \), a shock wave develops. Due to higher velocity waves moving in front of lower velocity waves, a number of compression waves merge into ‘one steep finite pressure wave’ or shock wave [2] (page 117). The pressure change across a shock wave is rapid. Unlike subsonic flow, when an object is travelling in supersonic flow, it cannot be assumed that this flow is isentropic so the equation for subsonic flow is no longer valid. The flow can no longer be considered a reversible adiabatic (i.e. isentropic) process as the shock waves lead to discontinuities in properties of the flow (including velocity, pressure and density) due to the dissipation of energy [4] (page 116).

At this point, it is important to consider an additional property of the flow, namely entropy. Entropy is ‘a measure of the amount of energy which is unavailable to do work’ [7]. An alternative definition is that entropy is a measure of how much disorder occurs in a system. This definition makes it easier to see that entropy increases (or stays constant), since a natural, isolated system becomes more disordered over time (going from low disorder (low entropy) to high disorder (high entropy)). Note that previously, an adiabatic process was defined as a process where no heat is exchanged between a system and its surroundings. An alternative, equivalent description of an adiabatic process is that the ‘specific entropy at the moving particle remains constant’ [4] (page 5). This does not mean that entropy is constant throughout the fluid, in fact there is an increase in entropy across a shock wave because of the change in pressure [2] (page 117).

3.1.1 Normal Shock Waves

A normal shock wave is one which is normal i.e. perpendicular to the flow. The remainder of this chapter follows the notation in Figure 5. Properties with a subscript 1 denote the values of those properties upstream in front of the shock wave, whilst those with a subscript 2 refer to those downstream i.e. behind the shock wave. In general, the quantities upstream are known (or can be measured directly) whilst those downstream are the ones to be determined (in relation to those upstream). Similar to Section 2, the flow is considered to be steady, inviscid and adiabatic, though not isentropic, with no gravitational forces. The flow can be considered adiabatic because the shock wave itself does not do work and there is no heat addition to the system, so the total temperature remains constant i.e. there is no heat exchange which is one definition of adiabatic. Also assume that the flow is one-dimensional and that \( u_i = (U_i, 0, 0) \), \( U_i > 0 \ \forall i \).

Derivation of the Rankine-Hugoniot Equations:

Define a control volume, \( A \), over the shock wave. Inside this volume; momentum, mass continuity, energy and state equations define the flow [2] (page 120). By first considering the momentum flux (which is defined as the momentum flow per area i.e. the force due to the pressure change),
the following is obtained [8]:

\[-\rho_1 U_1^2 A + \rho_2 U_2^2 A - p_1 A + p_2 A = 0\]
\[\rho_1 U_1^2 + p_1 = \rho_2 U_2^2 + p_2\]  \hspace{1cm} (11)

Now considering mass flux [8]:

\[-\rho_1 U_1 A + \rho_2 U_2 A = 0\]
\[\rho_1 U_1 = \rho_2 U_2\] \hspace{1cm} (12)

By considering conservation of energy and noting that since the flow is adiabatic, the enthalpy is equal on either side of the shock wave, that is, the relationship regarding enthalpy described in Equation (5) from Section 2 can be rewritten as [9]:

\[h_1 + \frac{U_1^2}{2} = h_2 + \frac{U_2^2}{2}\] \hspace{1cm} (13)

And finally the Equation of State, describing ‘increase or conservation of entropy’ gives [4] (pages 121-124):

\[p_2 = \frac{\gamma - 1}{\gamma} \rho_2 h_2\] \hspace{1cm} (14)

Together, Equations (11), (12), (13), and (14) are called the Rankine-Hugoniot Shock Equations [10], [2] (page 125).

**Derivation of Prandtl’s Relation for Normal Shock Waves:**

From Equations (5) and (13), the following can be defined [10]:

\[h_1 + \frac{U_1^2}{2} = h_0 = h_2 + \frac{U_2^2}{2}\] \hspace{1cm} (15)

\[18\]
Also note that from the definition of enthalpy in Section 2.4 and the definition of the speed of sound:

\[
\frac{h_1(\gamma - 1)}{c_1^2} = \frac{\gamma - 1}{\gamma - 1}RT_1 (\gamma - 1) = 1
\]

This means that [10]:

\[
c_1^2 = h_1 (\gamma - 1) = \left( h_0 - \frac{U_1^2}{2} \right) (\gamma - 1)
\]  

The relationship is analogous for the properties behind the shock wave, and a similar expression is derived for \( c_2^2 \). The reason that \( c_2 \) differs from \( c_1 \) is due to the definition of the local speed of sound. It was defined as follows:

\[
c = \sqrt{\gamma RT} = \sqrt{\frac{\gamma p}{\rho}}
\]

Clearly, since the values of \( p \) and \( \rho \) differ across the shock wave, the value of \( c \) will also differ.

Now consider Equations (11) and (12) relating to conservation of mass and momentum. Together they can be combined to give:

\[
U_1 + \frac{p_1}{\rho_1 U_1} = U_2 + \frac{p_2}{\rho_2 U_2}
\]

Rearranging:

\[
U_1 - U_2 = \frac{p_2}{\rho_2 U_2} - \frac{p_1}{\rho_1 U_1}
\]

\[
U_1 - U_2 = \frac{c_2^2}{\gamma U_2} - \frac{c_1^2}{\gamma U_1}
\]

Using the definition of the local speed of sound, given in Equation (16):

\[
U_1 - U_2 = \frac{\left( h_0 - \frac{U_2^2}{2} \right) (\gamma - 1)}{\gamma U_2} - \frac{\left( h_0 - \frac{U_1^2}{2} \right) (\gamma - 1)}{\gamma U_1}
\]

\[
U_1 - U_2 = \frac{\gamma - 1}{\gamma} \left( \frac{h_0}{U_2} - \frac{U_2}{2} - \frac{h_0}{U_1} + \frac{U_1}{2} \right)
\]

\[
U_1 - U_2 = \frac{\gamma - 1}{\gamma} \left( h_0 \left( \frac{U_1 - U_2}{U_1 U_2} \right) + \frac{1}{2} (U_1 - U_2) \right)
\]

Multiplying both sides by \( \frac{1}{U_1 - U_2} \), squaring both sides and rearranging gives:

\[
\frac{(\gamma - 1)^2 h_0^2}{U_1^2 U_2^2} = \left( \frac{\gamma + 1}{2} \right)^2
\]

Then, using the two definitions of \( h_0 \) given in Equation (15), this becomes:
\[
(\gamma - 1)^2 \left( h_1 + \frac{U_1^2}{2} \right) \left( h_2 + \frac{U_2^2}{2} \right) \left( \frac{2}{\gamma + 1} \right)^{\frac{3}{2}} = U_1^2 U_2^2
\]
\[
(\gamma - 1)^2 \left( \frac{h_1 c_1^2 + M_1^2}{c_1^2} \right) \left( \frac{h_2 c_2^2 + M_2^2}{c_2^2} \right) \left( \frac{2}{\gamma + 1} \right)^{\frac{3}{2}} = M_1^2 M_2^2
\]
\[
\left( \frac{h_1 (\gamma - 1)}{c_1^2} + \frac{(\gamma - 1) M_1^2}{2} \right) \left( \frac{h_2 (\gamma - 1)}{c_2^2} + \frac{(\gamma - 1) M_2^2}{2} \right) \left( \frac{2}{\gamma + 1} \right)^{\frac{3}{2}} = M_1^2 M_2^2
\]

Then since \( \frac{h_1(\gamma - 1)}{c_1^2} = 1 \) (and analogously behind the shock wave) this gives:

\[
M_1^2 M_2^2 = \left( \frac{2}{\gamma + 1} \right)^{\frac{3}{2}} \left( 1 + \frac{\gamma - 1}{2} M_1^2 \right) \left( 1 + \frac{\gamma - 1}{2} M_2^2 \right)
\]

Rearranging this gives:

\[
\frac{M_1^2 (\gamma + 1)^2}{4 \left( 1 + \frac{\gamma - 1}{2} M_1^2 \right)} = \frac{\gamma - 1}{2} + \frac{1}{M_2^2}
\]

\[
\frac{1}{M_2^2} = \frac{(\gamma + 1)^2 M_1^2 - 2 (\gamma - 1) \left( 1 + \frac{\gamma - 1}{2} M_1^2 \right)}{4 + 2 (\gamma - 1) M_1^2}
\]


\[
M_2^2 = \frac{4 + 2 (\gamma - 1) M_1^2}{(\gamma + 1)^2 M_1^2 - 2 (\gamma - 1) \left( 1 + \frac{\gamma - 1}{2} M_1^2 \right)}
\]

\[
M_2^2 = \frac{4 \left( 1 + \frac{\gamma - 1}{2} M_1^2 \right)}{(\gamma + 1)^2 - (\gamma - 1)^2} M_1^2 - 2 (\gamma - 1)
\]

\[
M_2^2 = \frac{1 + \frac{\gamma - 1}{2} M_1^2}{\gamma M_1^2 - \frac{(\gamma - 1)^2}{2}}
\]

\[
M_2 = \sqrt{\frac{2 + (\gamma - 1) M_1^2}{2 \gamma M_1^2 - (\gamma - 1)}}
\]

This is Prandtl’s Relation for Normal Shock Waves which gives a \( M_2 \) as a function of \( M_1 \).

**The Non-Existence of Shock Waves in Subsonic Flow:**

From Prandtl’s relation it is easy to see that if the flow in front of the shock wave is supersonic i.e. if \( M_1 > 1 \) then the flow behind it will be subsonic since \( M_2 < 1 \). (Also note that if \( M_1 = 1 \) then \( M_2 = 1 \).)

Prandtl’s relation can also be used to prove that shock waves do not appear in subsonic flow. Consider \( M_1 < 1 \) i.e. subsonic flow and assume a shock wave has formed. From Prandtl’s relation it suggests that \( M_2 > 1 \). This seems counter-intuitive as subsonic flow should not
become supersonic. This would mean that entropy would decrease over the shock wave. This is contradictory as entropy can only increase (or remain constant). Therefore there are no shock waves in subsonic flow [2] (page 132).

3.1.2 Oblique Shock Waves

Oblique shock waves are those which are, as the name suggests, at an oblique angle (not \( \frac{\pi}{2} \)) to the flow. Similar to normal shock waves, the values of properties of the flow (including pressure and density) undergo large changes across an oblique shock wave [2] (page 154).

As with normal shock waves, the Rankine-Hugoniot equations can also be formed for oblique shock waves. It is important to note that when the equivalent Rankine-Hugoniot Shock Equations are derived for oblique shock waves, they are equal to the equations derived for normal shock waves. This is true (and expected) because the quantities in the equations ‘are not affected by the moving coordinate system adopted to transform a normal shock into an oblique shock’ [2] (page 163). For this reason, oblique shock waves are not considered in depth in this project, as it is not necessary in deriving a relationship between pressure and velocity in relation to Pitot tubes in uniform flow.

3.1.3 Bow Shock Waves

A bow shock is so-called due to its shape. They are found in front of blunt objects with a curved leading edge (when the fluid in front is travelling at supersonic speeds) for example planetary bow shocks [12], early spacecraft and Pitot tubes. Bow shocks are ‘normal locally to the flow, just ahead of the stagnation point’ [8].

In general, due to the non-linear shape of a bow shock, analysing the properties of the flow before and after the shock in one-dimension is not sufficient and a more in-depth analysis is required.

3.2 Pitot Tubes and Bow Shocks

As mentioned above, given the rounded shape of a Pitot tube, a bow shock will form in front of it [13], as the aircraft travels at supersonic speed. In this section, a simple Pitot tube along with a surface perforation (see Figure 1, Point B) is considered. This allows a relationship between the pressures before and after the shock wave to be described more easily as the surface perforation is placed such that it is in front of the shock wave [2], (page 134). Across the bow shock, there is a steep pressure change due to the shock wave and, as such, the pressures differ. Since Pitot tubes are placed such that they are parallel to the flow this means that the bow shock in front of the Pitot tube is ‘normal at the stagnation streamline’ (and hence ‘symmetrical on both sides of the stagnation streamline’) [2], (page 135). This means that the one-dimensional relationships
derived for normal shock waves (in Section 3.1.1) can be used in relation to Pitot tubes.

In the following derivations, $p_{02}$ is the pressure behind the shock wave (measured by the manometer attached to the Pitot tube).

The aim of this section is to derive an equation which describes the relationship between $p_{02}$ and the pressure in front of the shock wave $p_1$. From the earlier derivations in Section 2.5, the following relationship holds immediately behind the shock wave [13]:

\[ p_{02} = \frac{p_1}{\gamma + 1} + \frac{\gamma \rho_1}{\gamma - 1} \]
\[ \frac{p_0}{p_2} = \left( 1 + \frac{\gamma - 1}{2} M_2^2 \right)^{\frac{\gamma}{\gamma - 1}} \]  

This is from the equation derived for subsonic, compressible flow. It is valid because, as discussed in Section 3.1.1, if \( M_1 > 1 \) then \( M_2 < 1 \) i.e. the flow behind the shock wave is subsonic.

### 3.3 Derivation of the Rayleigh Supersonic Pitot Equation

Whilst Prandtl’s Relation for Normal Shock Waves is useful because it gives an expression between the Mach numbers before and after the shock wave, in practice \( M_2 \) is unknown and the aim is to calculate \( M_1 \). The information that is known are the two pressure measurements from the Pitot tube (\( p_0 \) and \( p_1 \)), so a relationship between \( M_1 \) and these two pressures is needed. In order to derive an expression relating the stagnation pressure (measured by the manometer attached to Pitot tube) to the pressure in front of the shock wave, an expression relating the pressure in front of the shock wave to the pressure behind it must first be derived, in particular an expression for the ratio \( \frac{p_2}{p_1} \) will be derived [10] (page 125-126).

Using the definition of Mach number and properties of an ideal gas, the following can be defined:

\[ \rho_1 U_1^2 = \gamma p_1 M_1^2 \]

Similarly,

\[ \rho_2 U_2^2 = \gamma p_2 M_2^2 \]

By conservation of momentum (Equation (11)) this means that:

\[ p_1 + \gamma p_1 M_1^2 = p_2 + \gamma p_2 M_2^2 \]

Rearranging and using the expression derived for \( M_2 \) in Equation (17):

\[
\begin{align*}
\frac{p_2}{p_1} &= \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \\
\frac{p_2}{p_1} &= \frac{1 + \gamma M_1^2}{1 + \gamma \left( \frac{1 + (\gamma - 1) M_1^2}{\gamma M_1^2 - (\gamma - 1)} \right)} \\
\frac{p_2}{p_1} &= \frac{(1 + \gamma M_1^2) (2 \gamma M_1^2 - (\gamma - 1))}{2 \gamma M_1^2 - (\gamma - 1) + \gamma (2 + (\gamma - 1))} \\
\frac{p_2}{p_1} &= \frac{(1 + \gamma M_1^2) (2 \gamma M_1^2 - (\gamma - 1))}{(\gamma + 1) + \gamma (\gamma + 1) M_1^2} \\
\frac{p_2}{p_1} &= \frac{2 \gamma M_1^2 - (\gamma - 1)}{\gamma + 1} \\
\frac{p_2}{p_1} &= \frac{2 \gamma M_1^2 - (\gamma - 1)}{\gamma + 1}
\end{align*}
\]
Therefore the ratio is defined as [10]:

\[
\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1} (M_1^2 - 1)
\]  

(19)

Therefore:

\[
\begin{align*}
\frac{p_0_2}{p_1} &= \frac{p_0_2}{p_2} \frac{p_2}{p_1} \\
\frac{p_0_2}{p_1} &= \frac{p_0_2}{p_2} \left( 1 + \frac{2\gamma}{\gamma + 1} (M_1^2 - 1) \right) \\
\frac{p_0_2}{p_1} &= \frac{p_0_2}{p_2} \left( \frac{1 - \gamma}{\gamma + 1} + 2\gamma M_1^2 \right)
\end{align*}
\]

This can be combined with Equation (18) to get the following relationship:

\[
\frac{p_0_2}{p_1} = \left( 1 + \frac{\gamma - 1}{2} M_2^2 \right)^{\frac{\gamma - 1}{2}} \left( \frac{1 - \gamma}{\gamma + 1} + 2\gamma M_1^2 \right)
\]

Now using the Prandtl's Relation for relationship between \(M_1\) and \(M_2\) derived in Section 3.1.1. (Equation (17)) :

\[
\begin{align*}
\frac{p_0_2}{p_1} &= \left( 1 + \frac{\gamma - 1}{2} M_1^2 \right)^{\frac{\gamma - 1}{2}} \left( \frac{1 - \gamma}{\gamma + 1} + 2\gamma M_1^2 \right) \\
\frac{p_0_2}{p_1} &= \left( 1 + \frac{\gamma - 1}{2} \frac{(\gamma - 1)}{M_1^2} \right)^{\frac{\gamma - 1}{2}} \left( \frac{1 - \gamma}{\gamma + 1} + 2\gamma M_1^2 \right) \\
\frac{p_0_2}{p_1} &= \left( \frac{\gamma + 1}{4\gamma M_1^2 - 2(\gamma - 1)} \right)^{\frac{\gamma - 1}{2}} \left( \frac{1 - \gamma}{\gamma + 1} + 2\gamma M_1^2 \right)
\end{align*}
\]

Equation (20) is the Rayleigh Supersonic Pitot Equation [13].

The Rayleigh Supersonic Pitot Equation gives an expression for the Mach number of the fluid, \(M_1\), in terms of the two pressures which can be measured by the Pitot tube (and the perforation measuring the static pressure). This equation is written in terms of Mach number from which the velocity, \(U\), can be derived if the speed of sound is known. Due to the complexity of the equation, an explicit solution for \(M_1\) cannot be derived and numerical techniques need to be used to solve the equation.

3.3.1 Example: Solution of the Rayleigh Supersonic Pitot Equation

As discussed above, there is no explicit solution for \(M_1\) (and hence for \(U\)) and instead a numerical solution is required to find the Mach number. Consider the example of an aircraft flying
at 10000m (around 33,000ft) where \( p_1 \) is 26436Pa and \( p_0 \), as measured by the Pitot tube is 191657Pa.

Then by the Rayleigh Supersonic Pitot Equation:

\[
\frac{191657}{26436} = \left( \frac{(\gamma + 1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma - 1)} \right)^{\frac{1}{\gamma - 1}} \left( \frac{(1 - \gamma) + 2\gamma M_1^2}{\gamma + 1} \right)
\]

Then substituting in \( \gamma = 1.4 \):

\[
\left( \frac{5.76 M_1^2}{5.6 M_1^2 - 0.8} \right)^{3.5} \left( \frac{-0.4 + 2.8 M_1^2}{2.4} \right) - \frac{191657}{26436} = 0
\]

**Newton-Raphson Method:**

The Newton-Raphson method is as follows [14] (page 328): To find the root of an equation, \( f \), using some initial approximation (often found graphically) \( x_0 \), the following algorithm is applied until it converges:

\[
x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)}
\]

In order for the Newton-Raphson method to be applied to the above problem, the derivative of the function must be found. Defining \( F \) to be the Rayleigh Supersonic Pitot Equation, rearranged to equal zero:

\[
F(M_1) = \left( \frac{(\gamma + 1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma - 1)} \right)^{\frac{1}{\gamma - 1}} \left( \frac{(1 - \gamma) + 2\gamma M_1^2}{\gamma + 1} \right) - \frac{p_0}{p_1}
\]

Then the following can be derived:

\[
\frac{dF(M_1)}{dM_1} = \frac{d}{dM_1} \left( \left( \frac{(\gamma + 1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma - 1)} \right)^{\frac{1}{\gamma - 1}} \left( \frac{(1 - \gamma) + 2\gamma M_1^2}{\gamma + 1} \right) - \frac{p_0}{p_1} \right)
\]

\[
\frac{dF(M_1)}{dM_1} = \left( \left( \frac{(\gamma + 1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma - 1)} \right)^{\frac{1}{\gamma - 1}} \left( \frac{4\gamma M_1}{\gamma + 1} \right) \right) + \left( \frac{(1 - \gamma) + 2\gamma M_1^2}{\gamma + 1} \right) \left( \frac{\gamma}{\gamma - 1} \left( \frac{(\gamma + 1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma - 1)} \right)^{\frac{1}{\gamma - 1}} \right) \left( \frac{(1 - \gamma) M_1}{2\gamma M_1^2 - \gamma + 1} \right)^2
\]

Note that this derivative does not depend on \( p_0 \) or \( p_1 \) and so is valid for all supersonic flow. However, for the specific example above, it reduces to:

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\[ \frac{dF(M_1)}{dM_1} = \left( \frac{49M_1^3}{18} - \frac{7M_1}{18} \right) + \left( \frac{504M_1(0.4 - 2.8M_1^2)}{24(7M_1^2 - 1)^2} \right) \left( \frac{5.76M_1^2}{5.6M_1^2 - 0.8} \right)^{2.5} \]

Using the Python code in Appendix 3, with an initial estimate of \( M_1 = 2 \) and convergence to 3 decimal places, a value of 2.249 is returned for \( M_1 \). Since this value is greater than one, this is consistent with supersonic flow.

Additionally, rearranging Equation (4) to express the Mach number in terms of the pressure ratio (when the flow is assumed to be incompressible and subsonic) and similarly for Equation (10) (when the flow is assumed to be compressible and subsonic), Figure 8 shows the Mach number calculated for each of Equations (4) and (8), as well as the value from the Rayleigh Supersonic Pitot Equation, as the ratio between the two pressures \( \left( \frac{p_0}{p_1} \right) \) increases. Note that in Figure 8, the Mach number calculated from the Rayleigh Supersonic Pitot Equation is only shown when \( M > 1 \) (because it was derived based on a discontinuity in the flow which does not occur when \( M < 1 \)).

For the specific example above (where the Pitot tube pressure is 191657Pa), Equation (4) would calculate a value of 2.928 for the Mach number, whilst Equation (10) gives 1.460. This clearly shows that, of the three equations considered, only the Rayleigh Supersonic Pitot Equation is valid to use when the flow is supersonic. Unsurprisingly, if either of the two equations relating to subsonic flow are used when the flow is actually supersonic, the error is large. Interestingly, one of them (assuming incompressibility) gives a larger Mach number than the ‘true’ value calculated by the Rayleigh Supersonic Pitot Equation, whereas the other (assuming compressibility) gives a value lower than that of the true value.

![Figure 8: Calculation of Mach Number](image)

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4 Conclusion

4.1 Summary

The main objective of this project was to investigate the relationship between pressure and velocity for one-dimensional flows at Mach numbers above and below the threshold value 1. Three main equations have been derived relating to: subsonic incompressible flow (where it has been shown that, for the subsonic case, use of the incompressible equation for compressible gases can lead to calculation errors as large as 25%); subsonic compressible flow; and supersonic compressible flow. Different types of shock waves and their properties have been discussed with a set of defining equations having been derived for Normal shock waves and applied to derive the Rayleigh Supersonic Pitot Equation. These equations have been applied to the flow through a Pitot tube on an aircraft, to determine how the airspeed is calculated and, in the case of supersonic flow, using a numerical method to do so.

4.2 Further Study

4.2.1 Pitot Tube Shape

Delta and swept wing aircraft have, as the name suggests, triangular or swept-back wings. This has the benefit of reducing the wing span of the aircraft. This reduced wing span means that swept wings are often used in aircraft designed for supersonic flight because, as a shock wave develops along the surface of the wing, the drag increases. This drag increases as the wing span increases, thus a shorter wing span means less drag. [15]

A similar technique can be used on Pitot tubes to reduce the drag generated by the bow shock which develops in front of it.

Figure 9: Shock Wave around a Swept Pitot Tube

It would be interesting to further investigate the idea of swept Pitot tubes to conclude how the angle and general shape of a Pitot tube affects the flow. Although Pitot tubes are very small in comparison to the whole aircraft (and therefore their shock waves are comparatively small), it would also be of interest to analyse by how much the drag from the shock wave in front of
the Pitot tube contributes to the drag of the aircraft as a whole (and whether it can be deemed negligible).

### 4.2.2 Pitot Tube Blockage

One of the main reasons for Pitot Tube failure is a blockage in the tube, often due to insects or the formation of ice crystals. This means that the pressure at the stagnation point is not the same as the pressure recorded by the manometer attached to it, resulting in calculations of the indicated airspeed (IAS) and Mach number which are incorrect. Most commercial aircraft are usually equipped with four Pitot tubes so that, if the average airspeed appears to be dropping, the readings from individual Pitot tubes can indicate if one of the four readings is incorrect i.e. if one of them may be blocked.

Unreliable airspeeds mean that pilots are manoeuvring aircraft based on incorrect information and for Air France Flight 447 travelling from Rio de Janeiro to Paris, in June 2009, this proved fatal. ‘The sudden drop in the measured airspeeds, likely due to the obstruction of the Pitot probes by ice crystals, caused autopilot and autothrust disconnection’ [16]. This inaccurate pressure measurement, combined with the fact that the co-pilots had ‘not received any training, at high altitude, in the ‘Unreliable IAS’ procedure and manual aircraft handling’ [16] caused the aircraft to stall and, 4 minutes 23 seconds later, the aircraft crashed into the Atlantic Ocean. There were no survivors. Following the accident, the model of Pitot tube which was being used by the aircraft manufacturer was changed to one which was deemed to be more reliable in preventing ice-over.

A potential extension to the above work would be to consider the effect of complete Pitot tube blockage on the calculations of airspeed, and the resulting change if it were to become unblocked during flight.
References & Bibliography

References


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Bibliography

Books:
Waves in Fluids, Lighthill, James, Cambridge University Press, 1978,
Fundamentals of Compressible Flow (SI Units), Yahya, S.M., Halsted Press, 1982

e-Books:
Convergence and Applications of Newton-Type Iterations, Argyros, Ioannis K., Springer, 2008,
Incompressible and Low-Speed Flows, Drikakis, Dimitris and Rider, William, Springer, 2005,
Compressibility, Turbulence and High Speed Flow, Gatski, Thomas B. and Bonnet, Jean-Paul, Elsevier Ltd, 2013, 2nd Edition,
General Aviation Aircraft Design: Applied Methods and Procedures Gudmundsson, Snorri, Elsevier Inc, 2014,
Introduction to Aircraft Aeroelasticity and Loads, Wright, Jan R. and Cooper, Jonathan E.,
Websites:
Appendix

1. R Code for $(z_A - z_B) = 0$ Error Graph

```r
p<-seq(100000,101324,by=0.01)
error=(sqrt((2*((101325-p)/1.225))+0.0981))-(sqrt(2*((101325-p)/1.225)))
plot(p,error,type="l",xlab="Pressure (Pa)",ylab="Error",lwd=2)
abline(h=c(0,0.01,0.02,0.03),
v=c(100000,100200,100400,100600,100800,101000,101200),lty=2)
```

2. R Code for Incompressibility Assumption Percentage Error Graph

```r
M<-seq(0,1,by=0.01)
ga=1.4
Y=(((((1+((ga-1)/2)*M^2))^(ga/(ga-1)))-1)/((ga/2)*M^2)-1)
plot(M,Y,type="l",xlab="Mach Number (M)",ylab="Error",lwd=2)
abline(h=c(0,0.05,0.10,0.15,0.20,0.25), v=c(0.0,0.2,0.4,0.6,0.8,1.0),lty=2)
```


```python
from numpy import *

def F(M,pr):
    F = (((5.76*M**2)/(5.6*M**2 - 0.8))**3.5)*((2.8*M**2 - 0.4)/(2.4)) - (pr)
    return F

def dFdM(M):
    dFdM = (((49*M**3)-(7*M))/18) + ((8.4*M - 58.8*M**3)/((7*M**2 - 1)**2))*
    (((5.76*M**2)/(5.6*M**2 - 0.8))**2.5)
    return dFdM

def root(M0,pr,F,dFdM,err,maxit):
    error = 1.0
    n = 0
    while ((abs(error) > abs(err)) and (n < maxit)):
        n = n+1
        error = - F(M,pr)/dFdM(M,pr)
        M = M + error
    return M

M0=2
err = 1.0e-6
maxit = 30
pr= 191657/26436
root(M0,pr,F,dFdM,err,maxit)
```
import matplotlib.pyplot as plt
import math
gamma = 1.4
def subincMach(pr):
    M = math.sqrt((2.0/gamma)*(pr-1))
    return M

P = arange(1.0,6.0,0.1)
Mlist = []
for i in range(len(P)):
    k = subincMach(P[i])
    Mlist.append(k)

def subcomMach(pr):
    M = math.sqrt(((2.0/(gamma-1))*(1 - ((pr)**((1-gamma)/gamma)))))
    return M

Mlist2 = []
for i in range(len(P)):
    k = subcomMach(P[i])
    Mlist2.append(k)

def supMach(pr):
    M = froot(M0,pr,F,dFdM,err,nmax)
    return M

P2 = arange(2.0,6.0,0.01)
Mlist3 = []
for i in range(len(P2)):
    k = supMach(P2[i])
    Mlist3.append(k)

plt.plot(P,Mlist,label="Eqn (4) - Subsonic Incompressible Flow")
plt.plot(P,Mlist2,label="Eqn (10) - Subsonic Compressible Flow")
plt.plot(P2,Mlist3,label="Eqn (20) - Supersonic Flow")
plt.legend(loc="top left")
plt.show()